# 1 Differential Equations

***EXAMPLE 1-1:***

$$\begin{aligned} 0 &= B - 8B^4 \\ &= B\left(1 - 8B^3\right) \end{aligned}$$

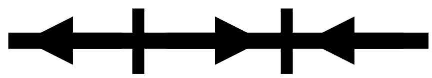
$$B^* = 0$$

$$1 - 8B^3 = 0$$

$$8B^3 = 1$$

$$B^3 = \frac{1}{8}$$

$$B^* = \frac{1}{2}$$



*B*=0* *B*=1/2*

*unstable* *stable*

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$$B^3 = \frac{1}{8}$$

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## 1.2 Differential Equations: Your Turn!

***Solutions:***

a)

$$0 = \frac{5P^2}{4+P^2} - P$$

$$P = \frac{5P^2}{4+P^2}$$

$$4P + P^3 = 5P^2$$

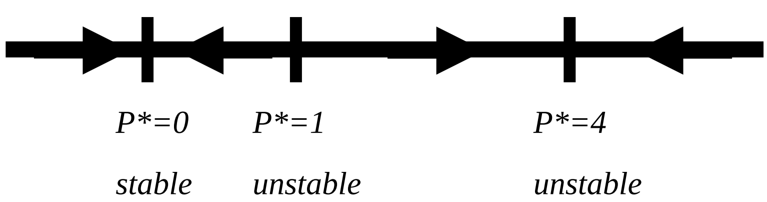
$$4P - 5P^2 + P^3 = 0$$

$$P\left(4 - 5P + P^2\right) = 0$$

$$P(P-1)(P-4) = 0$$

$$P^* = 0, 1, 4$$

| | P < 0 | 0 < P < 1 | 1 < P < 4 | P > 4 |
|---|---|---|---|---|
| dP/dt | + | – | + | – |
| Phase diagram | Arrow to right | Arrow to left | Arrow to right | Arrow to left |



$$\frac{5P^2}{4+P^2} - P$$

$$\left(\frac{dP}{dt}\right)' = \frac{10P\left(4+P^2\right) - 2P\left(10P^2\right)}{\left(4+P^2\right)^2} - 1$$

$$= \frac{40P + 10P^3 - 20P^3}{\left(4+P^2\right)^2} - 1$$

$$= \frac{40P - 10P^3}{\left(4+P^2\right)^2} - 1$$

$$\left.\left(\frac{dP}{dt}\right)'\right|_{P^*=0} = \frac{40(0) - 10(0)^3}{\left[4+(0)^2\right]^2} - 1 = -1 \rightarrow \textit{stable}$$

$$\left.\left(\frac{dP}{dt}\right)'\right|_{P^*=1} = \frac{40(1) - 10(1)^3}{\left[4+(1)^2\right]^2} - 1 = \frac{1}{5} \rightarrow \textit{unstable}$$

$$\left.\left(\frac{dP}{dt}\right)'\right|_{P^*=4} = \frac{40(4) - 10(4)^3}{\left[4+(4)^2\right]^2} - 1 = -\frac{11}{5} \rightarrow \textit{stable}$$

The results from the phase line diagram and the Stability Theorem are the same. Cool.

# 2 Linear Algebra

***EXAMPLE 2-1:***

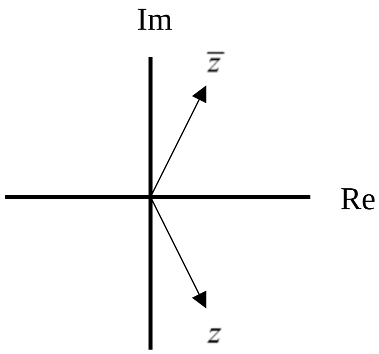
$$
\begin{aligned}
z - w &= (3-2i)-(6+i) \\
&= -3 - i \\
zw &= (3-2i)(6+i) \\
&= 18 + 3i - 12i - 2i^2 \\
&= 18 - 9i - 2(-1) \\
&= 18 - 9i + 2 \\
&= 20 - 9i
\end{aligned}
$$

***EXAMPLE 2-2:***

$$\bar{z} = 2 + 3i$$

$$|z| = \sqrt{a^2 + b^2} = \sqrt{2^2 + 3^2} = \sqrt{13}$$

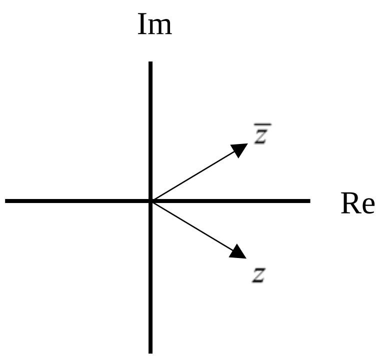
$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{2+3i}{13} = \frac{2}{13} + \frac{3}{13}i$$



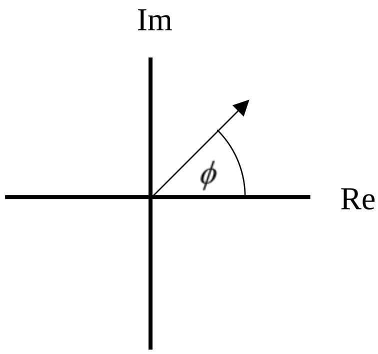

$$\bar{z} = 4 - 2i$$

$$|z| = \sqrt{a^2 + b^2} = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{4-2i}{20} = \frac{4}{20} - \frac{2}{20}i = \frac{1}{5} - \frac{1}{10}i$$



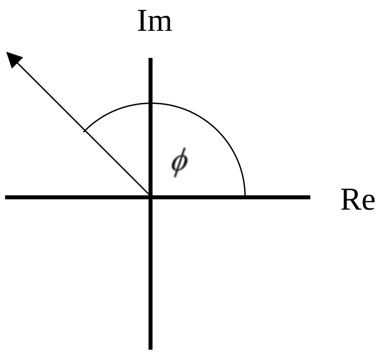
***EXAMPLE 2-3:***




$$r = |z| = \sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\phi = \frac{\pi}{4}$$

$$z = \sqrt{2}e^{i\frac{\pi}{4}} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$



***EXAMPLE 2-4:***

*Matrix form:*

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -1 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & -1 & 1 & 0 \end{array}\right]$$

***<u>EXAMPLE 2-5:</u>***

*Matrix form:*

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & -2 & 2 & 4 \\ -5 & 6 & -3 & -7 \end{array}\right]$$

*Gauss time!*

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & -2 & 2 & 4 \\ -5 & 6 & -3 & -7 \end{array}\right] \xrightarrow[5R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -8 & -1 & 1 \\ 0 & 16 & 2 & -2 \end{array}\right]$$

$$\xrightarrow{R_2/-8} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1/8 & -1/8 \\ 0 & 16 & 2 & -2 \end{array}\right]$$

$$\xrightarrow[-16R_2+R_3]{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & \boxed{3/4} & 5/4 \\ 0 & 1 & \boxed{1/8} & -1/8 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right]$$

*Solution type:* Infinite solutions.

$$x = \frac{5}{4} - \frac{3}{4}t$$
$$y = -\frac{1}{8} - \frac{1}{8}t$$
$$z = t$$

***EXAMPLE 2-6:***

*Matrix form:*

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 10 \\ 3 & 1 & 6 & 9 \\ 1 & 1 & 1 & 4 \end{array}\right]$$

*Gauss time!*

$$\left[\begin{array}{ccc|c} \underline{1} & 2 & 4 & 10 \\ 3 & 1 & 6 & 9 \\ 1 & 1 & 1 & 4 \end{array}\right] \xrightarrow[-R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} \underline{1} & 2 & 4 & 10 \\ 0 & -5 & -6 & -21 \\ 0 & -1 & -3 & -6 \end{array}\right]$$

$$\xrightarrow{swap\ R_2, R_3} \left[\begin{array}{ccc|c} \underline{1} & 2 & 4 & 10 \\ 0 & -1 & -3 & -6 \\ 0 & -5 & -6 & -21 \end{array}\right]$$

$$\xrightarrow{R_2/-1} \left[\begin{array}{ccc|c} \underline{1} & 2 & 4 & 10 \\ 0 & \underline{1} & 3 & 6 \\ 0 & -5 & -6 & -21 \end{array}\right]$$

$$\xrightarrow[5R_2+R_3]{-2R_2+R_1} \left[\begin{array}{ccc|c} \underline{1} & 0 & -2 & -2 \\ 0 & \underline{1} & 3 & 6 \\ 0 & 0 & 9 & 9 \end{array}\right]$$

$$\xrightarrow{R_3/9} \left[\begin{array}{ccc|c} \underline{1} & 0 & -2 & -2 \\ 0 & \underline{1} & 3 & 6 \\ 0 & 0 & \underline{1} & 1 \end{array}\right]$$

$$\xrightarrow[-3R_3+R_2]{2R_3+R_1} \left[\begin{array}{ccc|c} \underline{1} & 0 & 0 & 0 \\ 0 & \underline{1} & 0 & 3 \\ 0 & 0 & \underline{1} & 1 \end{array}\right]$$

*Solution type:* Unique solution.

$x_1 = 0$

$x_2 = 3$

$x_3 = 1$

***EXAMPLE 2-7:***

*Gauss!*

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & a-3 & b-2 \end{array}\right]$$

$$\xrightarrow[-2R_2+R_3]{-R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-5 & b-4 \end{array}\right] \longleftarrow \boxed{\text{Stop here!}}$$

*No solutions:*

Need a zombie row. This happens if $a = 5, b \neq 4$.

*Unique solution:*

Need a non-zombie, non-dead row. This happens if $a \neq 5, b$ can be any real number.

*Infinite solutions:*

Need a dead row. This happens if $a = 5, b = 4$.

***EXAMPLE 2-8:***

$$A = \begin{bmatrix} 1 & 5 \\ 6 & 4 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 2 \\ 3 & 3 \\ 5 & 7 \\ 1 & 2 \end{bmatrix}$$

$$\begin{aligned} 3A - 2B &= 3\begin{bmatrix} 1 & 5 \\ 6 & 4 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & 2 \\ 3 & 3 \\ 5 & 7 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 15 \\ 18 & 12 \\ 6 & 9 \\ 9 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 4 \\ 6 & 6 \\ 10 & 14 \\ 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 11 \\ 12 & 6 \\ -4 & -5 \\ 7 & -4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} 2B^T - A^T &= 2\begin{bmatrix} 1 & 5 \\ 6 & 4 \\ 2 & 3 \\ 3 & 0 \end{bmatrix}^T - \begin{bmatrix} 0 & 2 \\ 3 & 3 \\ 5 & 7 \\ 1 & 2 \end{bmatrix}^T \\ &= 2\begin{bmatrix} 1 & 6 & 2 & 3 \\ 5 & 4 & 3 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 3 & 5 & 1 \\ 2 & 3 & 7 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 12 & 4 & 6 \\ 10 & 8 & 6 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 3 & 5 & 1 \\ 2 & 3 & 7 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 9 & -1 & 5 \\ 8 & 5 & -1 & -2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} 5\left(B^T\right)^T &= 5B \\ &= 5\begin{bmatrix} 0 & 2 \\ 3 & 3 \\ 5 & 7 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 10 \\ 15 & 15 \\ 25 & 35 \\ 5 & 10 \end{bmatrix} \end{aligned}$$

***EXAMPLE 2-9:***

$$AB = \begin{bmatrix} 20 \\ 6 \\ 32 \end{bmatrix}$$

$$CA = \begin{bmatrix} 7 & 11 & 8 \end{bmatrix}$$

***EXAMPLE 2-10:***

$$\left[\begin{array}{ccc|ccc} \underline{1} & 0 & 2 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|ccc} \underline{1} & 0 & 2 & 1 & 0 & 0 \\ 0 & 4 & -2 & -2 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1/4} \left[\begin{array}{ccc|ccc} \underline{1} & 0 & 2 & 1 & 0 & 0 \\ 0 & \underline{1} & -1/2 & -1/2 & 1/4 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{ccc|ccc} \underline{1} & 0 & 2 & 1 & 0 & 0 \\ 0 & \underline{1} & -1/2 & -1/2 & 1/4 & 0 \\ 0 & 0 & 1/2 & -1/2 & -1/4 & 1 \end{array}\right]$$

$$\xrightarrow{2R_3} \left[\begin{array}{ccc|ccc} \underline{1} & 0 & 2 & 1 & 0 & 0 \\ 0 & \underline{1} & -1/2 & -1/2 & 1/4 & 0 \\ 0 & 0 & \underline{1} & -1 & -1/2 & 2 \end{array}\right]$$

$$\xrightarrow[1/2R_3+R_2]{-2R_3+R_1} \left[\begin{array}{ccc|ccc} \underline{1} & 0 & 0 & 3 & 1 & -4 \\ 0 & \underline{1} & 0 & -1 & 0 & 1 \\ 0 & 0 & \underline{1} & -1 & -1/2 & 2 \end{array}\right]$$

***EXAMPLE 2-11:***

$$\det A = (2)(-3)-(1)(6) = -6-6 = -12$$
$$\det B = (0)(0)-(3)(1) = 0-3 = -3$$
$$\det C = (2)(0)-(3)(0) = 0-0 = 0$$

A and B are invertible. C is not invertible.

***EXAMPLE 2-12:***

$$A = \begin{bmatrix} \boxed{1} & \boxed{0} & \boxed{5} \\ 1 & 1 & 2 \\ 3 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} \boxed{+} & \boxed{-} & \boxed{+} \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{aligned} \det A &= 1\left[(1)(4)-(2)(0)\right]-0\left[(1)(4)-(2)(3)\right]+5\left[(1)(0)-(1)(3)\right] \\ &= 1(4-0)-0(4-6)+5(0-3) \\ &= 4-0-15 \\ &= -11 \end{aligned}$$

$$B = \begin{bmatrix} 4 & 4 & 1 \\ 1 & 1 & 6 \\ \boxed{0} & \boxed{0} & \boxed{7} \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ \boxed{+} & \boxed{-} & \boxed{+} \end{bmatrix}$$

$$\begin{aligned} \det B &= 0\left[(4)(6)-(1)(1)\right]-0\left[(4)(6)-(1)(1)\right]+7\left[(4)(1)-(4)(1)\right] \\ &= 0(24-1)-0(24-1)+7(4-4) \\ &= 0-0+0 \\ &= 0 \end{aligned}$$

A is invertible. B is not invertible.

***EXAMPLE 2-13:***

$$\lambda I - A = \begin{bmatrix} \lambda - 13 & \boxed{0} & 15 \\ 3 & \boxed{\lambda - 4} & -9 \\ -5 & \boxed{0} & \lambda + 7 \end{bmatrix} \quad \begin{bmatrix} + & \boxed{-} & + \\ - & \boxed{+} & - \\ + & \boxed{-} & + \end{bmatrix}$$

$$\det(\lambda I - A) = -0\det\begin{bmatrix} 3 & -9 \\ -5 & \lambda + 5 \end{bmatrix} + (\lambda - 4)\det\begin{bmatrix} \lambda - 13 & 15 \\ -5 & \lambda + 7 \end{bmatrix} - 0\det\begin{bmatrix} \lambda - 13 & 15 \\ 3 & \lambda + 7 \end{bmatrix}$$

$$= \cancel{-0\det\begin{bmatrix} 3 & -9 \\ -5 & \lambda + 5 \end{bmatrix}} + (\lambda - 4)\det\begin{bmatrix} \lambda - 13 & 15 \\ -5 & \lambda + 7 \end{bmatrix} \cancel{-0\det\begin{bmatrix} \lambda - 13 & 15 \\ 3 & \lambda + 7 \end{bmatrix}}$$

$$= (\lambda - 4)\left[(\lambda - 13)(\lambda + 7) - (15)(-5)\right]$$

$$= (\lambda - 4)\left(\lambda^2 - 6\lambda - 91 + 75\right)$$

$$= (\lambda - 4)\left(\lambda^2 - 6\lambda - 16\right)$$

$$\det(4I - A) = (4 - 4)\left[4^2 - 6(4) - 16\right] = 0$$

$\therefore \lambda = 4$ is an eigenvalue of $A$

$$\lambda^2 - 6\lambda - 16 = 0$$

$$(\lambda - 8)(\lambda + 2) = 0$$

$$\lambda = -2, 8$$

$\lambda = -2, 4, 8$ are all the eigenvalues of $A$

$$4I-A=\left[\begin{array}{ccc|c}-9&0&15&0\\3&0&-9&0\\-5&0&11&0\end{array}\right]$$

$$\xrightarrow{R_1\ swap\ R_2}\left[\begin{array}{ccc|c}3&0&-9&0\\-9&0&15&0\\-5&0&11&0\end{array}\right]$$

$$\xrightarrow{R_1/3}\left[\begin{array}{ccc|c}1&0&-3&0\\-9&0&15&0\\-5&0&11&0\end{array}\right]$$

$$\xrightarrow[5R_1+R_3]{9R_1+R_2}\left[\begin{array}{ccc|c}1&0&-3&0\\0&0&-12&0\\0&0&-4&0\end{array}\right]$$

$$\xrightarrow{R_2/-12}\left[\begin{array}{ccc|c}1&0&-3&0\\0&0&1&0\\0&0&-4&0\end{array}\right]$$

$$\xrightarrow[4R_2+R_3]{3R_2+R_1}\left[\begin{array}{ccc|c}1&\boxed{0}&0&0\\0&\boxed{0}&1&0\\0&\boxed{0}&0&0\end{array}\right]$$

$x=0$

$y=t$

$z=0$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=t\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$\begin{bmatrix}0\\1\\0\end{bmatrix}$ is an eigenvector for $\lambda=4$

***EXAMPLE 2-14:***

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & 2 \\ -1 & \lambda - 3 \end{bmatrix}$$

$$\begin{aligned} \det(\lambda I - A) &= (\lambda - 1)(\lambda - 3) - (2)(-1) \\ &= \lambda^2 - 4\lambda + 3 + 2 \\ &= \lambda^2 - 4\lambda + 5 \end{aligned}$$

Can't factor. Use quadratic formula.

$$\begin{aligned} \lambda &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{4 \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} \\ &= \frac{4 \pm \sqrt{-4}}{2} \\ &= \frac{4 \pm \sqrt{4}i}{2} \\ &= \frac{4 \pm 2i}{2} \\ &= 2 \pm i \end{aligned}$$

$$\boxed{\lambda = 2+i}$$

$$\left[\begin{array}{cc|c} 1+i & 2 & 0 \\ -1 & -1+i & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \text{ swap } R_2} \left[\begin{array}{cc|c} -1 & -1+i & 0 \\ 1+i & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_1/-1} \left[\begin{array}{cc|c} \underline{1} & 1-i & 0 \\ 1+i & 2 & 0 \end{array}\right]$$

$$\xrightarrow{(-1-i)R_1+R_2} \left[\begin{array}{cc|c} \underline{1} & \boxed{1-i} & 0 \\ 0 & \boxed{0} & 0 \end{array}\right]$$

$$x+(1-i)y=0$$
$$y=t$$

$$x=(-1+i)t$$
$$y=t$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} -1+i \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$eigenvector\ \#1 = \cos(\beta t)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \sin(\beta t)\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \cos t\begin{bmatrix} -1 \\ 1 \end{bmatrix} - \sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\cos t - \sin t \\ \cos t \end{bmatrix}$$

$$eigenvector\ \#2 = \sin(\beta t)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \cos(\beta t)\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \sin t\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \cos t\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin t + \cos t \\ \sin t \end{bmatrix}$$

***EXAMPLE 2-15:***

$$A=\begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix}$$

$$\lambda I-A=\begin{bmatrix}\lambda-2 & -3\\ -1 & \lambda-4\end{bmatrix}$$

$$\begin{aligned}\det(\lambda I-A)&=(\lambda-2)(\lambda-4)-(-3)(-1)\\ &=\lambda^2-6\lambda+8-3\\ &=\lambda^2-6\lambda+5\\ &=(\lambda-5)(\lambda-1)\end{aligned}$$

$$\lambda=1,5$$

$$\boxed{\lambda=1}$$

$$I-A=\left[\begin{array}{cc|c}-1 & -3 & 0\\ -1 & -3 & 0\end{array}\right]$$

$$\xrightarrow{-R_1}\left[\begin{array}{cc|c}\underline{1} & 3 & 0\\ -1 & -3 & 0\end{array}\right]$$

$$\xrightarrow{R_1+R_2}\left[\begin{array}{cc|c}\underline{1} & \boxed{3} & 0\\ 0 & \boxed{0} & 0\end{array}\right]$$

$$x+3y=0$$

$$y=t$$

$$x=-3t$$

$$y=t$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=t\begin{bmatrix}-3\\ 1\end{bmatrix}$$

$\boxed{\lambda = 5}$

$$I - A = \left[\begin{array}{cc|c} 3 & -3 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1/3} \left[\begin{array}{cc|c} \underline{1} & -1 & 0 \\ -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1+R_2} \left[\begin{array}{cc|c} \underline{1} & \boxed{-1} & 0 \\ 0 & \boxed{0} & 0 \end{array}\right]$$

$$x - y = 0$$
$$y = t$$

$$x = t$$
$$y = t$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

General solution: $\begin{bmatrix} x \\ y \end{bmatrix} = C_1 e^{t} \begin{bmatrix} -3 \\ 1 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Sub in initial conditions:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = C_1 e^{0} \begin{bmatrix} -3 \\ 1 \end{bmatrix} + C_2 e^{5(0)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$1 = -3C_1 + C_2$$
$$2 = C_1 + C_2$$

$$C_2 = 2 - C_1$$
$$-3C_1 + (2 - C_1) = 1$$
$$-4C_1 = -1$$
$$C_1 = 1/4$$
$$C_2 = 7/4$$

Particular solution: $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} e^{t} \begin{bmatrix} -3 \\ 1 \end{bmatrix} + \frac{7}{4} e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

## 2.6 Linear Algebra: Your Turn!

***Solutions:***

a)

i)

$$\begin{aligned} z+w &= (9+i)+(4-i) \\ &= 13 \end{aligned}$$

ii)

$$\begin{aligned} 2zw &= (9+i)(4-i) \\ &= 36+4i-9i-i^2 \\ &= 36-5i-(-1) \\ &= 36-5i+1 \\ &= 37-5i \end{aligned}$$

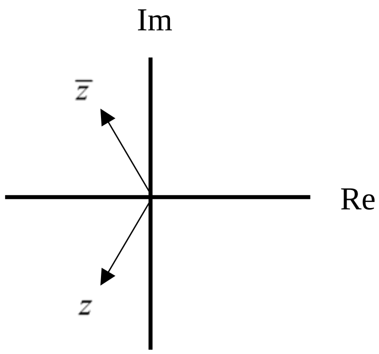
b)

i)

$$\bar{z} = -1 + 2i$$

$$|z| = \sqrt{a^2 + b^2} = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{-1+2i}{5} = \frac{-1}{5} + \frac{2}{5}i$$

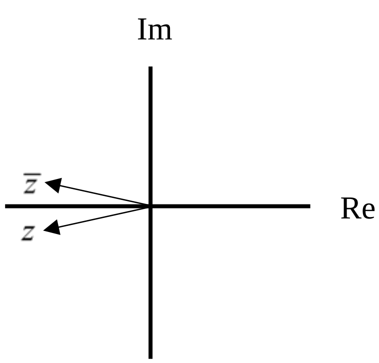



ii)

$$\bar{z} = -3 + \frac{i}{2}$$

$$|z| = \sqrt{a^2 + b^2} = \sqrt{(-3)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{37}{4}} = \frac{1}{2}\sqrt{37}$$

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{-3+\frac{i}{2}}{37/4} = \frac{-12}{37} + \frac{2}{37}i$$



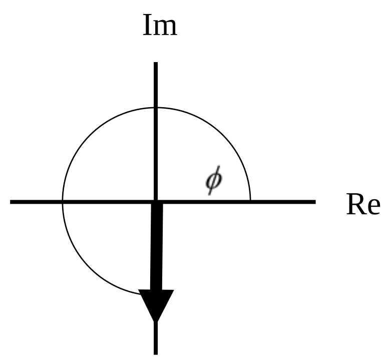
c)

i)

$$r = |z| = \sqrt{a^2 + b^2} = \sqrt{0^2 + (-5)^2} = \sqrt{25} = 5$$

$$\phi = \frac{3\pi}{2}$$

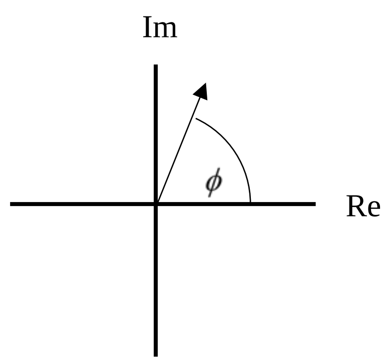
$$z = 5e^{i\frac{3\pi}{2}} = 5\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$




You can figure out $\phi$ from the graph.

It is ¾ of a full circle, which is $\frac{3\pi}{2}$ in radians.

ii)




$$r = |z| = \sqrt{a^2 + b^2} = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$\phi = \arctan\left(\frac{3}{1}\right) \cong 1.25$$

$$z = \sqrt{10}e^{1.25i} = \sqrt{10}\left(\cos 1.25 + i\sin 1.25\right)$$

d) *Matrix form:*

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 5 \\ 2 & 6 & 5 & 2 & 16 \\ 0 & 0 & 1 & 2 & 6 \end{array}\right]$$

*Gauss time!*

$$\left[\begin{array}{cccc|c} \underline{1} & 3 & 2 & 0 & 5 \\ 2 & 6 & 5 & 2 & 16 \\ 0 & 0 & 1 & 2 & 6 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cccc|c} \underline{1} & 3 & 2 & 0 & 5 \\ 0 & 0 & \underline{1} & 2 & 6 \\ 0 & 0 & 1 & 2 & 6 \end{array}\right]$$

$$\xrightarrow[-R_2+R_3]{-2R_2+R_1} \left[\begin{array}{cccc|c} \underline{1} & \boxed{3} & 0 & \boxed{-4} & -7 \\ 0 & \boxed{0} & \underline{1} & \boxed{2} & 6 \\ 0 & \boxed{0} & 0 & \boxed{0} & 0 \end{array}\right] \longleftarrow \text{Dead row}$$

*Solution type:* Infinite solutions. Two free variables (parameters).

Information from the matrix:

$$x + 3y - 4w = -7$$
$$z + 2w = 6$$
$$y = s$$
$$w = t$$

Rewrite in terms of free variables:

$$x = -7 - 3s + 4t$$
$$y = s$$
$$z = 6 - 2t$$
$$w = t$$

e)

*Matrix form:*

$$\left[\begin{array}{ccc|c} 3 & -2 & 9 & 14 \\ 1 & -1 & 2 & 4 \\ 2 & -4 & a & b \end{array}\right]$$

*Gauss!*

$$\left[\begin{array}{ccc|c} 3 & -2 & 9 & 14 \\ 1 & -1 & 2 & 4 \\ 2 & -4 & a & b \end{array}\right] \xrightarrow{\text{swap } R_1, R_2} \left[\begin{array}{ccc|c} \underline{1} & -1 & 2 & 4 \\ 3 & -2 & 9 & 14 \\ 2 & -4 & a & b \end{array}\right]$$

$$\xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left[\begin{array}{ccc|c} \underline{1} & -1 & 2 & 4 \\ 0 & \underline{1} & 3 & 2 \\ 0 & -2 & a-4 & b-8 \end{array}\right]$$

$$\xrightarrow[2R_2+R_3]{R_2+R_1} \left[\begin{array}{ccc|c} \underline{1} & 0 & 5 & 6 \\ 0 & \underline{1} & 3 & 2 \\ 0 & 0 & a+2 & b-4 \end{array}\right] \longleftarrow \boxed{\text{Stop here!}}$$

*No solutions:*

Need a zombie row. This happens if $a = -2, b \neq 4$.

*Unique solution:*

Need a non-zombie, non-dead row. This happens if $a \neq -2, b$ can be any real number..

*Infinite solutions:*

Need a dead row. This happens if $a = -2, b = 4$.

f)

$$5A+2B=5\begin{bmatrix}1&1&0\\2&5&1\\6&12&7\end{bmatrix}+2\begin{bmatrix}0&0&5\\5&1&9\\1&7&4\end{bmatrix}$$
$$=\begin{bmatrix}5&5&0\\10&25&5\\30&60&35\end{bmatrix}+\begin{bmatrix}0&0&10\\10&2&18\\2&14&8\end{bmatrix}$$
$$=\begin{bmatrix}5&5&10\\20&27&23\\32&74&43\end{bmatrix}$$

$$3B^T+4A^T=3\begin{bmatrix}0&0&5\\5&1&9\\1&7&4\end{bmatrix}^T+4\begin{bmatrix}1&1&0\\2&5&1\\6&12&7\end{bmatrix}^T$$
$$=3\begin{bmatrix}0&5&1\\0&1&7\\5&9&4\end{bmatrix}+4\begin{bmatrix}1&2&6\\1&5&12\\0&1&7\end{bmatrix}$$
$$=\begin{bmatrix}0&15&3\\0&3&21\\15&27&12\end{bmatrix}+\begin{bmatrix}4&8&24\\4&20&48\\0&4&28\end{bmatrix}$$
$$=\begin{bmatrix}4&23&27\\4&23&69\\15&31&40\end{bmatrix}$$

$$4\left(A^T\right)^T=4A$$
$$=4\begin{bmatrix}1&1&0\\2&5&1\\6&12&7\end{bmatrix}$$
$$=\begin{bmatrix}4&4&0\\8&20&4\\24&48&28\end{bmatrix}$$

g)

$$\left[\begin{array}{ccc|ccc} \underline{1} & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|ccc} \underline{1} & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2/2} \left[\begin{array}{ccc|ccc} \underline{1} & 0 & 1 & 1 & 0 & 0 \\ 0 & \underline{1} & -1/2 & -1/2 & 1/2 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|ccc} \underline{1} & 0 & 1 & 1 & 0 & 0 \\ 0 & \underline{1} & -1/2 & -1/2 & 1/2 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3/2} \left[\begin{array}{ccc|ccc} \underline{1} & 0 & 1 & 1 & 0 & 0 \\ 0 & \underline{1} & -1/2 & -1/2 & 1/2 & 0 \\ 0 & 0 & \underline{1} & 0 & -1/2 & 1/2 \end{array}\right]$$

$$\xrightarrow[1/2R_3+R_2]{-R_3+R_1} \left[\begin{array}{ccc|ccc} \underline{1} & 0 & 0 & 1 & 1/2 & -1/2 \\ 0 & \underline{1} & 0 & -1/2 & 1/4 & 1/4 \\ 0 & 0 & \underline{1} & 0 & -1/2 & 1/2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 1/2 & -1/2 \\ -1/2 & 1/4 & 1/4 \\ 0 & -1/2 & 1/2 \end{bmatrix}$$

h)

$$\det A = (5)(7) - (3)(0) = 35 - 0 = 35$$
$$\det B = (8)(4) - (2)(16) = 32 - 32 = 0$$
$$\det C = (1)(1) - (0)(0) = 1 - 0 = 1$$

A and C are invertible because their determinants are not zero.

B is not invertible because *det B = 0*.

i)

$$A = \begin{bmatrix} 6 & 1 & 1 \\ \boxed{0} & \boxed{1} & \boxed{7} \\ 1 & 4 & 8 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ \boxed{-} & \boxed{+} & \boxed{-} \\ + & - & + \end{bmatrix}$$

$$\begin{aligned} \det A &= -0\left[\cancel{(1)(8)-(1)(4)}\right] + 1\left[(6)(8)-(1)(1)\right] - 7\left[(6)(4)-(1)(1)\right] \\ &= 0 + 1(48-1) - 7(24-1) \\ &= 47 - 161 \\ &= -114 \end{aligned}$$

$$B = \begin{bmatrix} \boxed{5} & \boxed{5} & \boxed{5} \\ 4 & 1 & 7 \\ 1 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} \boxed{+} & \boxed{-} & \boxed{+} \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{aligned} \det B &= 5\left[(1)(2)-(7)(1)\right] - 5\left[(4)(2)-(7)(1)\right] + 5\left[(4)(1)-(1)(1)\right] \\ &= 5(2-7) - 5(8-7) + 5(4-1) \\ &= -25 - 5 + 15 \\ &= -15 \end{aligned}$$

A and B are invertible because their determinants are not zero.

j)

i)

$$\lambda I - A = \begin{bmatrix} \lambda-1 & -2 & \boxed{-1} \\ -6 & \lambda+1 & \boxed{0} \\ 1 & 2 & \boxed{\lambda+1} \end{bmatrix} \quad \begin{bmatrix} + & - & \boxed{+} \\ - & + & \boxed{-} \\ + & - & \boxed{+} \end{bmatrix}$$

$$\begin{aligned}
\det(\lambda I - A) &= -1\det\begin{bmatrix} -6 & \lambda+1 \\ 1 & 2 \end{bmatrix} - 0\det\begin{bmatrix} \lambda-1 & -2 \\ 1 & 2 \end{bmatrix} + (\lambda+1)\det\begin{bmatrix} \lambda-1 & -2 \\ -6 & \lambda+1 \end{bmatrix} \\
&= -1\det\begin{bmatrix} -6 & \lambda+1 \\ 1 & 2 \end{bmatrix} \cancel{-0\det\begin{bmatrix} \lambda-1 & -2 \\ 1 & 2 \end{bmatrix}} + (\lambda+1)\det\begin{bmatrix} \lambda-1 & -2 \\ -6 & \lambda+1 \end{bmatrix} \\
&= -1\det\begin{bmatrix} -6 & \lambda+1 \\ 1 & 2 \end{bmatrix} + (\lambda+1)\det\begin{bmatrix} \lambda-1 & -2 \\ -6 & \lambda+1 \end{bmatrix} \\
&= -\left[(-6)(2) - (\lambda+1)(1)\right] + (\lambda+1)\left[(\lambda-1)(\lambda+1) - (-2)(-6)\right] \\
&= -(-12 - \lambda - 1) + (\lambda+1)(\lambda^2 - 1 - 12) \\
&= 13 + \lambda + \lambda^3 + \lambda^2 - 13\lambda - 13 \\
&= \lambda^3 + \lambda^2 - 12\lambda
\end{aligned}$$

$$\begin{aligned}
\det(3I - A) &= (3)^3 + (3)^2 - 12(3) \\
&= 27 + 9 - 36 \\
&= 0
\end{aligned}$$

$\therefore \lambda = 3$ is an eigenvalue of $A$

$$\begin{aligned}
\det(0I - A) &= (0)^3 + (0)^2 - 12(0) \\
&= 0
\end{aligned}$$

$\therefore \lambda = 0$ is an eigenvalue of $A$

$$\begin{aligned}
\det(-4I - A) &= (-4)^3 + (-4)^2 - 12(-4) \\
&= -64 + 16 + 48 \\
&= 0
\end{aligned}$$

$\therefore \lambda = -4$ is an eigenvalue of $A$

ii)

$$3I - A = \left[\begin{array}{ccc|c} 2 & -2 & -1 & 0 \\ -6 & 4 & 0 & 0 \\ 1 & 2 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \text{ swap } R_3} \left[\begin{array}{ccc|c} \underline{1} & 2 & 4 & 0 \\ -6 & 4 & 0 & 0 \\ 2 & -2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[-2R_1+R_3]{6R_1+R_2} \left[\begin{array}{ccc|c} \underline{1} & 2 & 4 & 0 \\ 0 & 16 & 24 & 0 \\ 0 & -6 & -9 & 0 \end{array}\right]$$

$$\xrightarrow{R_2/16} \left[\begin{array}{ccc|c} \underline{1} & 2 & 4 & 0 \\ 0 & \underline{1} & 3/2 & 0 \\ 0 & -6 & -9 & 0 \end{array}\right]$$

$$\xrightarrow[6R_2+R_3]{-2R_2+R_1} \left[\begin{array}{ccc|c} \underline{1} & 0 & \boxed{1} & 0 \\ 0 & \underline{1} & \boxed{3/2} & 0 \\ 0 & 0 & \boxed{0} & 0 \end{array}\right]$$

$$x + z = 0$$
$$y + 3/2z = 0$$
$$z = t$$

$$x = -t$$
$$y = -3/2t$$
$$z = t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ -3/2 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ -3/2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 3$

k)

$$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ 8 & \lambda - 4 \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda(\lambda - 4) - (-1)(8)$$

$$= \lambda^2 - 4\lambda + 8$$

Can't factor. Use quadratic formula.

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{(-4)^2 - 4(1)(8)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{-16}}{2}$$

$$= \frac{4 \pm \sqrt{16}i}{2}$$

$$= \frac{4 \pm 4i}{2}$$

$$= 2 \pm 2i$$

$$\boxed{\lambda = 2+2i}$$

$$\left[\begin{array}{cc|c} 2+2i & -1 & 0 \\ 8 & -2+2i & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \text{ swap } R_2} \left[\begin{array}{cc|c} 8 & -2+2i & 0 \\ 2+2i & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1/8} \left[\begin{array}{cc|c} 1 & -1/4+i/4 & 0 \\ 2+2i & -1 & 0 \end{array}\right]$$

$$\xrightarrow{(-2-2i)R_1+R_2} \left[\begin{array}{cc|c} 1 & \boxed{-1/4+i/4} & 0 \\ 0 & \boxed{0} & 0 \end{array}\right]$$

$$x + (-1/4 + i/4)y = 0$$

$$y = t$$

$$x = (1 - i/4)t$$

$$y = t$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1-i/4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1/4 \\ 0 \end{bmatrix}$$

$$\textit{eigenvector } \#1 = \cos(\beta t)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \sin(\beta t)\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \cos t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} -1/4 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos t + \frac{1}{4}\sin t \\ \cos t \end{bmatrix}$$

$$\textit{eigenvector } \#2 = \sin(\beta t)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \cos(\beta t)\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \sin t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos t \begin{bmatrix} -1/4 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin t - \frac{1}{4}\cos t \\ \sin t \end{bmatrix}$$

l)

i)

$$A=\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}$$

$$\lambda I-A=\begin{bmatrix}\lambda-1 & -2\\ 0 & \lambda-3\end{bmatrix}$$

$$\det(\lambda I-A)=(\lambda-1)(\lambda-3)-(-2)(0)$$

$$=(\lambda-1)(\lambda-3)$$

$$\lambda=1,3$$

$$\boxed{\lambda=1}$$

$$I-A=\left[\begin{array}{cc|c}0 & -2 & 0\\ 0 & -2 & 0\end{array}\right]$$

$$\xrightarrow{R_1/-2}\left[\begin{array}{cc|c}0 & \underline{1} & 0\\ 0 & -2 & 0\end{array}\right]$$

$$\xrightarrow{2R_1+R_2}\left[\begin{array}{cc|c}\boxed{0} & \underline{1} & 0\\ \boxed{0} & 0 & 0\end{array}\right]$$

$$x=t$$

$$y=0$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=t\begin{bmatrix}1\\ 0\end{bmatrix}$$

$\boxed{\lambda = 3}$

$$I - A = \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1/2} \left[\begin{array}{cc|c} \underline{1} & \boxed{-1} & 0 \\ 0 & \boxed{0} & 0 \end{array}\right]$$

$$x - y = 0$$
$$y = t$$

$$x = t$$
$$y = t$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

General solution: $\begin{bmatrix} x \\ y \end{bmatrix} = C_1 e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Sub in initial conditions:

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = C_1 e^{0} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{3(0)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$3 = C_1 + C_2$$
$$1 = C_2$$

$$C_2 = 1$$
$$C_1 = 3 - C_2$$
$$= 3 - 1$$
$$= 2$$

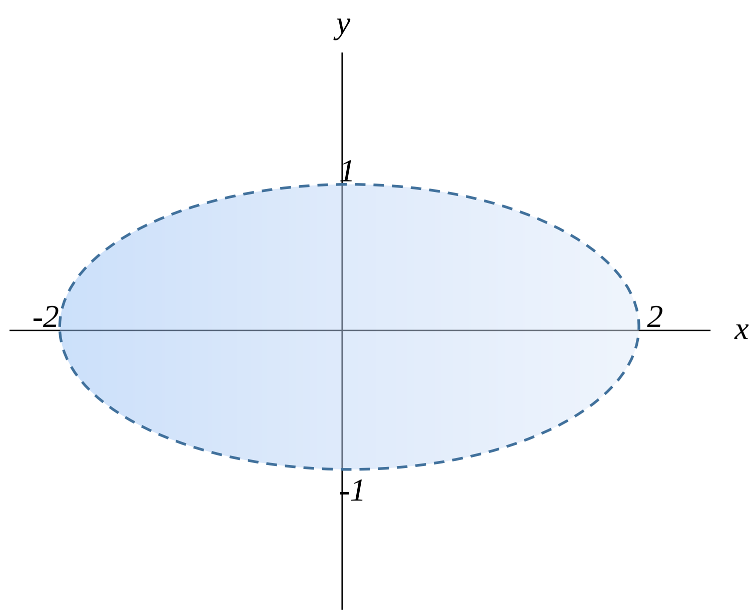
Particular solution: $\begin{bmatrix} x \\ y \end{bmatrix} = 2e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

# 3 Multivariable Calculus

***EXAMPLE 3-1:***

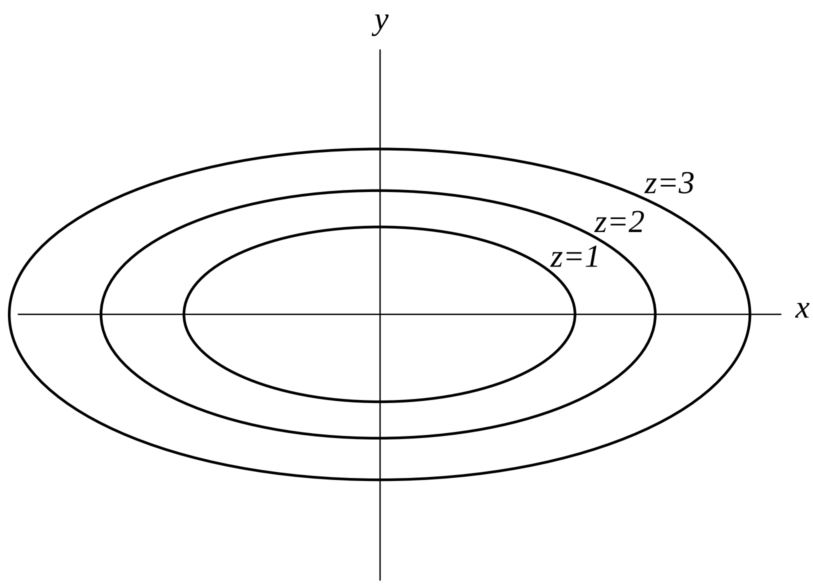
$$4 - x^2 - 4y^2 > 0$$
$$-x^2 - 4y^2 > -4$$
$$x^2 + 4y^2 < 4$$
$$\frac{x^2}{4} + y^2 < 1$$

$$D = \left\{ (x, y) \in R^2 \mid \frac{x^2}{4} + y^2 < 1 \right\}$$



***EXAMPLE 3-2:***

| Level | Resulting Formula | Level Curve (ie. written in "ellipse" form) |
|---|---|---|
| z = 1 | $x^2 + 4y^2 = 1$ | $x^2 + 4y^2 = 1$ |
| z = 2 | $x^2 + 4y^2 = 2$ | $\frac{x^2}{2} + 2y^2 = 1$ |
| z = 3 | $x^2 + 4y^2 = 3$ | $\frac{x^2}{3} + \frac{4y^2}{3} = 1$ |



## 3.3 Multivariable Calculus: Your Turn!

***Solutions:***

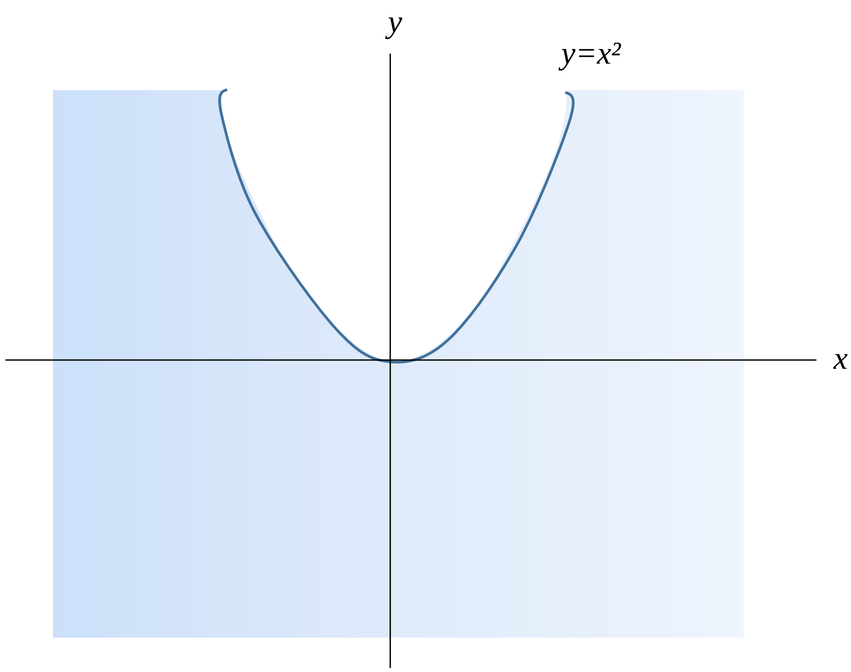
a)

i)

$$x^2 - y \geq 0$$
$$-y \geq -x^2$$
$$y \leq x^2$$

$$D = \{(x, y) \in R^2 \mid y \leq x^2\}$$

ii)



iii)

| Level | Resulting Formula | Level Curve (ie. solved for *y*) |
|---|---|---|
| z = 1 | $\sqrt{x^2 - y} = 1$ | $y = x^2 - 1$ |
| z = 2 | $\sqrt{x^2 - y} = 2$ | $y = x^2 - 4$ |
| z = 3 | $\sqrt{x^2 - y} = 3$ | $y = x^2 - 9$ |