Sample Problems for Midterm # 2
Calculus 1 MAT 1320 B

1. Find the derivative of the following functions.

a) $f(x) = \ln(\cos(x))$.

Solution:

$$f'(x) = -\frac{\sin(x)}{\cos(x)} = -\tan(x).$$

b) $f(x) = \arcsin(e^x)$.

Solution:

$$f'(x) = \frac{e^x}{\sqrt{1-e^{2x}}}.$$

c) $f(x) = (\tan(2^x))^{10}$.

Solution:

$$f'(x) = 10\left(\tan(2^x))\right)^9 \sec^2(2^x)\ln(2)2^x.$$

d) $f(x) = \sin(\log_3(e^x))$.

Solution:

$$f'(x0 = \cos(\log_3(e^x))\left(\frac{1}{\ln(3)e^x}\right)e^x = \frac{\cos(\log_3(e^x))}{\ln(3)}.$$

2. Find dy/dx.

a) $y = \sin\left(e^x - y^2\right)$.

Solution: Taking d/dx on both sides we get

$$y' = \cos\left(e^x - y^2\right)(e^x - 2yy').$$

Solving for y' we get

$$y' = \frac{e^x \cos\left(e^x - y^2\right)}{1 + 2y\cos(e^x - y^2)}.$$

b) $3xy^2 - xy = e^y$.

Solution: Taking d/dx on both sides we get

$$3y^2 + 6xyy' - y - xy' = e^y y'.$$

Solving for y' we get

$$y' = \frac{y - 3y^2}{6xy - x - e^y}.$$

c) $y = x^{\tan(x)-x^2}$.

Solution: Taking natural logarithm on both sides we get

$$\ln(y) = \big(\tan(x) - x^2\big)\ln(x).$$

Taking d/dx on both sides we get

$$\frac{y'}{y} = \big(\sec^2(x) - 2x\big)\ln(x) + \frac{\tan(x) - x^2}{x}.$$

Solving for y' we get

$$y' = \left(\big(\sec^2(x) - 2x\big)\ln(x) + \frac{\tan(x) - x^2}{x}\right) x^{\tan(x)-x^2}.$$

d) $y = (\ln(x))^{\ln(x)}$.

Solution: Taking natural logarithm on both sides we get

$$\ln(y) = \ln(x)\ \ln\left(\ln(x)\right).$$

Taking d/dx on both sides we get

$$\frac{y'}{y} = \frac{\ln\left(\ln(x)\right)}{x} + \ln(x)\frac{1}{x\ln(x)} = \frac{\ln\left(\ln(x)\right)}{x} + \frac{1}{x} = \frac{\ln\left(\ln(x)\right) + 1}{x}.$$

Solving for y' we get

$$y' = \left(\frac{\ln\left(\ln(x)\right) + 1}{x}\right)(\ln(x))^{\ln(x)}.$$

3. Linear Approximation.

a) Obtain the linear approximation of $f(x) = \ln(x-1)$ at $a = 2$. Then obtain an estimate of $\ln(1.2)$.

Solution: We have

$$f(2) = \ln(2-1) = \ln(1) = 0.$$

We also have that $f'(x) = 1/(x-1)$, so

$$f'(2) = \frac{1}{2-1} = 1.$$

Thus, the linear approximation to f at $a = 2$ is

$$f(x) = \ln(x-1) \approx L_2(x) = f'(2)(x-2) + f(2) = (x-2).$$

Now, we want to estimate $\ln(1.2)$ by using the linear approximation of $\ln(x-1)$, so we need to consider the case $1.2 = x - 1$. This implies $x = 1.2 + 1 = 2.2$, so we have

$$\ln(1.2) \approx L_2(2.2) = 2.2 - 2 = 0.2$$

b) Obtain the linear approximation of $f(x) = \sqrt{1 - x - x^3}$ at $a = 0$. Then obtain an estimate of $f(0.1)$.

Solution: We have

$$f(0) = \sqrt{1} = 1.$$

We also have that $f'(x) = \dfrac{-1 - 3x^2}{2\sqrt{1 - x - x^3}}$, so

$$f'(0) = \frac{-1}{2\sqrt{1}} = -\frac{1}{2}.$$

Thus, the linear approximation to f at $a = 0$ is

$$f(x) \approx L_0(x) = f'(0)(x - 0) + f(0) = -\frac{x}{2} + 1.$$

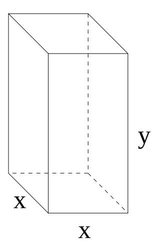
Therefore

$$f(0.1) \approx L_0(0.1) = -\frac{0.1}{2} + 1 = -0.05 + 1 = 0.95$$

4. Optimization.

a) A manufacturer wants to design an open box having a square base and a surface area of 108 cm^2. Find the dimensions that will produce a box with maximum volume.

Solution:



Let x be the length of any edge of the square base and let y be the height of the box. We want to optimize the function

$$V = x^2 y$$

under the condition

$$108 = x^2 + 4xy.$$

Now, the second equation implies

$$y = \frac{108 - x^2}{4x}.$$

Thus,

$$V(x) = x^2 \frac{108 - x^2}{4x} = \frac{108x - x^3}{4}, \qquad 0 < x < \sqrt{108}.$$

Their critical numbers are obtained via

$$0 = V'(x) = \frac{108 - 3x^2}{4}$$

whose solutions are

$$x = \pm 6.$$

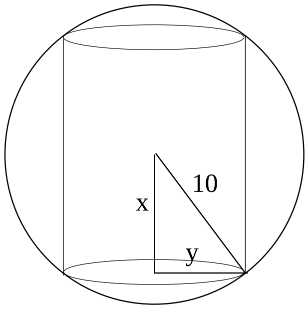
Only the value 6 can be used, we just need to see that at this value the volume has a maximum. Since

$$V''(6) = -\frac{3}{2}x_{\,|x=6} = -9,$$

we have that V has the maximum value at $x = 6$ and $y = 3$.

b) A right circular cylinder is inscribed in a sphere of radius 10. Find the largest possible volume of such a cylinder.

Solution:



Let x be half the height of the cylinder and let y be the radius of the base of the cylinder. The function we want to optimize is

$$V = \pi y^2(2x).$$

From the picture we can check that we have the relation

$$x^2 + y^2 = 100.$$

From the second equation we have

$$y^2 = 100 - x^2.$$

Thus,

$$V(x) = 2\pi x(100 - x^2) = 200\pi x - 2\pi x^3.$$

Their critical points are obtained via

$$0 = V'(x) = 200\pi - 6\pi x^2,$$

whose solutions are

$$x = \pm\frac{10}{\sqrt{3}}.$$

Only the value $\frac{10}{\sqrt{3}}$ can be used, we just need to see that at this value the volume has a maximum. Since

$$V''\left(\frac{10}{\sqrt{3}}\right) = -12\pi\left(\frac{10}{\sqrt{3}}\right) < 0,$$

we have that V has the maximum value at $x = 10/\sqrt{3}$ and $y = \sqrt{200/3}$.

5. Related Rates.

a) A spherical Balloon is being blown up. At a certain instant air enters the balloon so that the volume of the balloon increase at a rate of 50 cubic centimeters per second. At the same instant the balloon has a radius of 10 centimeters. How fast is the radius changing with time?

Recall that the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere.

Solution: From the formula for volume, taking d/dt we have that the rate of change of the volume at time t is

$$V' = 4\pi r^2 r'.$$

From this equation, at the given instant when $V' = 50$ and $r = 10$ we get

$$50 = 4\pi(100)r'.$$

Thus

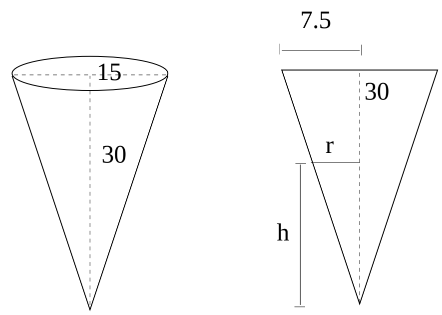
$$r' = \frac{50}{400\pi} = \frac{1}{8\pi} \quad \text{centimeters per second.}$$

at the given instant.

b) A thunderstorm is dropping into a conical tank at a rate of 5/4 inches^3 per hour. If the tank has diameter 15 inches and height 30 inches, at what rate is the water level rising when the water is 20 inches deep?

Recall that the volume of a cylinder is given by $V = \frac{1}{3}\pi r^2 h$, where r is the radius and h is the height.

Solution:



Let h be the height of the water level in the tank at time t, and let r be the radius of the top level of the water at time t. We want h' when $h = 20$.

Using similar triangles we have that

$$\frac{r}{h} = \frac{7.5}{30} = \frac{1}{4}$$

Then, $r = \dfrac{h}{4}$.

Thus, the volume of the tank is given by

$$V = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h = \frac{\pi h^3}{48}$$

After we differentiate with respect of t this equation we get

$$V' = \frac{3\pi h^2 h'}{48} = \frac{\pi h^2 h'}{16}$$

Using the information given in the problem we have the equation

$$\frac{5}{4} = \frac{\pi (20)^2 h'}{16}$$

Solving for h' we get

$$h' = \frac{1}{20\pi} \qquad \text{inches per hour.}$$

6. Consider a function $f(x)$ such that

$$f(x) = \ln\left(4 - x^2\right), \qquad f'(x) = \frac{-2x}{4 - x^2}, \qquad f''(x) = \frac{-2x^2 - 8}{(4 - x^2)^2}.$$

Find its domain, intercepts, asymptotes, intervals where f is increasing/decreasing, local maximums/minimums, and intervals where f is concave up/concave down. Then sketch the graph of f.

Solution: Domain: since natural logarithm is only defined for positive numbers, we need to solve

$$4 - x^2 > 0,$$

whose solution is $-2 < x < 2$.

Intercepts: y-intercept is given at $f(0) = \ln(4)$.
For x-intercepts we need to solve

$$0 = \ln(4 - x^2)$$

whose solutions are $x = \pm\sqrt{3}$.

Asymptotes: since f is only defined for $-2 < x < 2$, we do not have horizontal asymptotes.
Since $\ln(x)$ and $4 - x^2$ are continuous, $f(x)$ is continuous on $-2 < x < 2$. Now, because

$$\lim_{x \to -2^+} \ln(4 - x^2) = \lim_{u \to 0^+} \ln(u) = -\infty,$$

and

$$\lim_{x \to 2^-} \ln(4 - x^2) = \lim_{u \to 0^+} \ln(u) = -\infty,$$

we conclude that $f(x)$ has vertical asymptote at $x = -2$ and $x = 2$.

Intervals where f is increasing/decreasing: The solutions of

$$0 = f'(x) = \frac{-2x}{4 - x^2}$$

are obtained by solving

$$0 = -2x,$$

whose solution is $x = 0$. Now, for $-2 < x < 0$ we have that f' is the quotient of positive expressions, so $f' > 0$ on $-2 < x < 0$.
Thus, f is increasing on $-2 < x < 0$.

For, $0 < x < 2$ we have that f' is the quotient of a negative expression by a positive expression, so $f' < 0$ on $0 < x < 2$.
Thus, f is decreasing on $0 < x < 2$.

Since f is continuous, from the work above we have that f has a maximum at $x = 0$.

Intervals where f is concave up/concave down: The solutions of

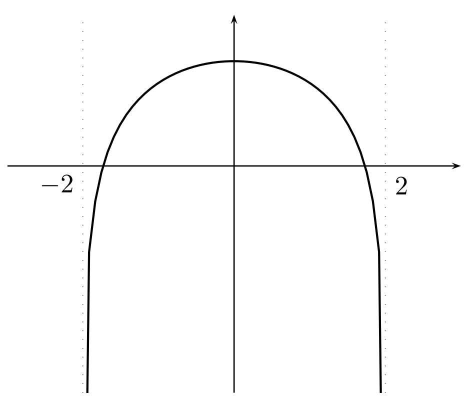
$$0 = f''(x) = \frac{-2x^2 - 8}{(4 - x^2)^2}$$

are obtained by solving

$$0 = -2x^2 - 8.$$

But such equation is equivalent to $x^2 = -2$, which has no solutions. This means that f is always either concave up or concave down. Now, since the denominator of f'' is always positive, but the nominator of f'' is always negative, we conclude that f'' is always negative. Thus, f is always concave down.

Putting all this information together we have that the graph of f is

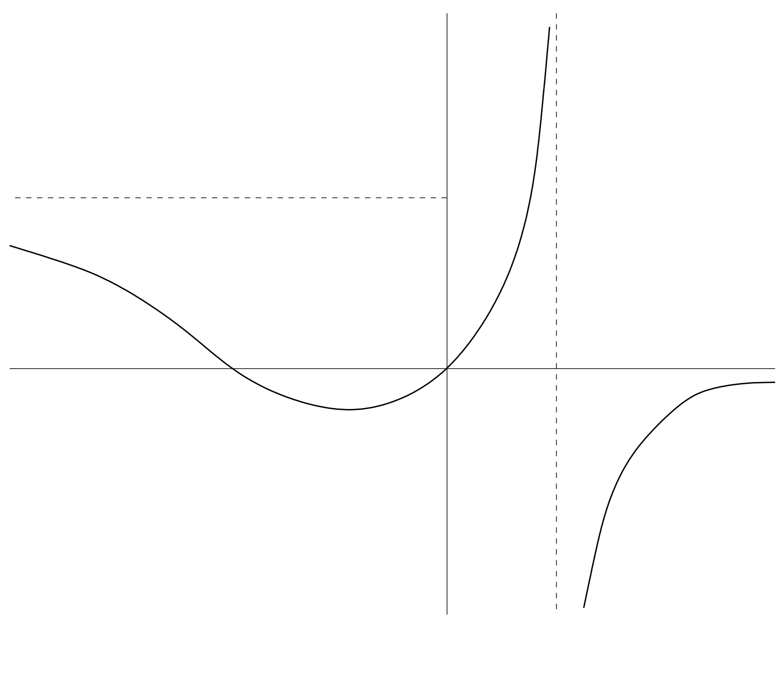


$f(x) = \ln(4 - x^2)$

7. Sketch the graph of a function $f(x)$ with the following properties.

- Domain: $(-\infty, 1) \cup (1, \infty)$.
- y-intercept at $y = 0$.
- x-intercepts at $x = -3, 0$.
- $\lim_{x \to 1^-} f(x) = \infty$, and $\lim_{x \to 1^+} f(x) = -\infty$.
- $\lim_{x \to -\infty} f(x) = 2$, and $\lim_{x \to \infty} f(x) = 0$.
- f is increasing on $(-2, 1)$ and $(1, \infty)$.
- f is decreasing on $(-\infty, -2)$.
- f is concave up on $(-3, 1)$.
- f is concave down on $(-\infty, -3)$ and $(1, \infty)$.

Solution:



8. Find the following limits.

a) $\lim_{x\to 0} \frac{3^x - 7^x}{x}$.

Solution:

$$\lim_{x\to 0} \frac{3^x - 7^x}{x} \quad \overset{=}{L'H} \quad \lim_{x\to 0} \frac{\ln(3)3^x - \ln(7)7^x}{1} = \ln(3) - \ln(7).$$

b) $\lim_{x\to 0^+} (\cos(x))^{1/x^2}$.

Solution: Consider

$$\lim_{x\to 0^+} (\cos(x))^{1/x^2} = \lim_{x\to 0^+} e^{\ln\left((\cos(x))^{1/x^2}\right)} = \lim_{x\to 0^+} e^{\frac{\ln(\cos(x))}{x^2}}$$

Since

$$\lim_{x\to 0^+} \frac{\ln(\cos(x))}{x^2} \quad \overset{=}{L'H} \quad \lim_{x\to 0^+} \frac{-\tan(x)}{2x} \quad \overset{=}{L'H} \quad \lim_{x\to 0^+} \frac{-\sec^2(x)}{2} = -\frac{1}{2}$$

we have that

$$\lim_{x\to 0^+} (\cos(x))^{1/x^2} = \lim_{x\to 0^+} e^{\frac{\ln(\cos(x))}{x^2}} = \lim_{u\to -1/2} e^u = e^{-1/2} = \frac{1}{\sqrt{e}}.$$

c) $\lim_{x\to 0^+} x^{x^2}$.

Solution: Consider

$$\lim_{x\to 0^+} x^{x^2} = \lim_{x\to 0^+} e^{\ln\left(x^{x^2}\right)} = \lim_{x\to 0^+} e^{x^2\ln(x)}$$

Since

$$\lim_{x\to 0^+} x^2\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/x^2} \quad \overset{=}{L'H} \quad \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} -\frac{x^3}{2x} = -\frac{1}{2}\lim_{x\to 0^+} x^2 = 0$$

we have that

$$\lim_{x\to 0^+} x^{x^2} = \lim_{x\to 0^+} e^{x^2\ln(x)} = \lim_{u\to 0} e^u = 1.$$