VERSION 1 SOLUTIONS

Please record your answers in this table.

Question	1	2	3	4	5	6	7	8	9	10
Answer	D	A	F	F	A	B	E	F	A	D

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1. Let π be the plane parallel to the vector $(1, 0, 3)$ such that π passes through the points $(0, 1, 1)$ and $(-3, 5, 2)$. Which of the following is a point-normal equation for π?

- [A] $3x + 2y + z = 3$
- [B] $4x + 3y - 2z = 1$
- [C] $5x - 3y - 2z = 7$
- [D] $6x + 5y - 2z = 3$
- [E] $5x + 4y - z = 3$
- [F] $2x - 9y + 6 = 4$

We need the normal of Π and a point on Π.
In order to get the normal vector, take the cross product of two direction vectors on Π.
Use $\underset{\sim}{u} = (1, 0, 3)$ and $\underset{\sim}{v} = (-3-0, 5-1, 2-1) = (-3, 4, 1)$.
Then $\underset{\sim}{n} = \underset{\sim}{u} \times \underset{\sim}{v} = (-12, -10, 4)$

$\begin{matrix} 0 & 3 & 1 & 0 \\ 4 & 1 & -3 & 4 \end{matrix}$
$(0\cdot1 - 4\cdot3,\ 3\cdot(-3) - 1\cdot1,\ 1\cdot4 - (-3)\cdot0)$

So an equation for Π is
$-12x - 10y + 4z = d$.
We can use the point $(0, 1, 1)$ to find d:
$-12\cdot0 - 10\cdot1 + 4\cdot1 = \boxed{-6 = d}$

So Π is given by $-12x - 10y + 4z = -6$ or, dividing by -2, it's $6x + 5y - 2z = 3$.

2. Let L be the line in $\mathbb{R}^3$ containing $(1, -1, 2)$ such that L is parallel to the two planes $x + 3y + 2z = 0$ and $4x - y - z = 1$. Which of the following is a set of parametric equations for L?

- [A] $x = 1 - t,\ y = -1 + 9t,\ z = 2 - 13t,\ t \in \mathbb{R}$
- [B] $x = -1 + 4t,\ y = -1 - 4t,\ z = 2 + 5t,\ t \in \mathbb{R}$
- [C] $x = t,\ y = -t,\ z = 2t,\ t \in \mathbb{R}$
- [D] $x = 1 - t,\ -1 + 9t,\ z = 2 + 13t,\ t \in \mathbb{R}$
- [E] $x = -1 - t,\ y = 1,\ z = -13t,\ t \in \mathbb{R}$
- [F] $x = -1 + 3t,\ y = -1 - t,\ z = 2 + 4t,\ t \in \mathbb{R}$

The normal of each of the given planes is orthogonal to L, so we can find a direction vector for L by crossing these.
$\underset{\sim}{n}_1 = (1, 3, 2)$ $\underset{\sim}{n}_2 = (4, -1, -1)$

$\underset{\sim}{n}_1 \times \underset{\sim}{n}_2 = (-1, 9, -13)$

$\begin{matrix} 3 & 2 & 1 & 3 \\ -1 & -1 & 4 & -1 \end{matrix}$

A line is defined by a point and a direction vector. We use the point $(1, -1, 2)$ and direction vector $(-1, 9, -13)$ to write
$x = 1 - t,\ y = -1 + 9t,\ z = 2 - 13t$ for parameter $t \in \mathbb{R}$.

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3. Let $\mathbf{u} = (-2, -3, 1)$, $\mathbf{v} = (2, 1, 0)$, and $\mathbf{w} = (1, 0, 1)$. What is the cosine of the angle between $(\mathbf{v} \times \mathbf{w})$ and $(\mathbf{u} \times \mathbf{v})$?

- [A] $\frac{1}{14}$
- [B] $-\frac{1}{\sqrt{21}}$.
- [C] $\frac{2}{21}$.
- [D] $\frac{\sqrt{2}}{\sqrt{21}}$
- [E] $-\frac{1}{\sqrt{14}}$
- [F] $-\frac{3}{\sqrt{14}}$

"Cosine" indicates that we will use the dot product to write $(\underset{\sim}{v}\times\underset{\sim}{w})\cdot(\underset{\sim}{u}\times\underset{\sim}{v}) = \|\underset{\sim}{v}\times\underset{\sim}{w}\|\cdot\|\underset{\sim}{u}\times\underset{\sim}{v}\|\cdot\cos\theta$.

That is, $\cos\theta = \dfrac{(\underset{\sim}{v}\times\underset{\sim}{w})\cdot(\underset{\sim}{u}\times\underset{\sim}{v})}{\|\underset{\sim}{v}\times\underset{\sim}{w}\|\cdot\|\underset{\sim}{u}\times\underset{\sim}{v}\|}$. (*)

$\underset{\sim}{v}\times\underset{\sim}{w} = (1,-2,-1)$ $\quad$ $\underset{\sim}{u}\times\underset{\sim}{v} = (-1,2,4)$ $\quad$ $(\underset{\sim}{v}\times\underset{\sim}{w})\cdot(\underset{\sim}{u}\times\underset{\sim}{v}) = 1(-1)-2(2$

$= -9$

1 0 2 1 $\quad$ -3 1 -2 -3

0 1 1 0 $\quad$ 1 0 2 1

Then $\|\underset{\sim}{v}\times\underset{\sim}{w}\| = \sqrt{1^2+(-2)^2+(-1)^2} = \sqrt{6}$

and $\|\underset{\sim}{u}\times\underset{\sim}{v}\| = \sqrt{(-1)^2+2^2+4^2} = \sqrt{21}$

Substitute these into (*): $\cos\theta = \dfrac{-9}{\sqrt{6}\cdot\sqrt{21}}$

We can simplify until we get one of the given choices.

$$\frac{-9}{\sqrt{6}\cdot\sqrt{21}} = \frac{-9}{\sqrt{2}\sqrt{3}\sqrt{3}\sqrt{7}} = \frac{-9}{\sqrt{2}\cdot 3\cdot\sqrt{7}} = \frac{-3}{\sqrt{2}\cdot\sqrt{7}} = \frac{-3}{\sqrt{14}}.$$

4. Let $\mathbf{u} = (1, 2, 1)$ and $\mathbf{v} = (-7, 4, 1)$. What is the orthogonal projection of $\mathbf{v}$ along $\mathbf{u}$?

- [A] $(-2, -4, -2)$
- [B] $(-\frac{1}{3}, -\frac{2}{3}, -\frac{1}{3})$
- [C] $(-\frac{1}{6}, -\frac{1}{3}, -\frac{1}{6})$
- [D] $(2, 4, 2)$
- [E] $(\frac{1}{6}, \frac{1}{3}, \frac{1}{6})$
- [F] $(\frac{1}{3}, \frac{2}{3}, \frac{1}{3})$

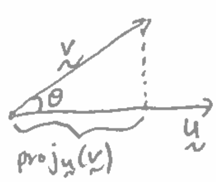
Recall the formula

$$\text{proj}_{\underset{\sim}{u}}(\underset{\sim}{v}) = \frac{\underset{\sim}{v}\cdot\underset{\sim}{u}}{\underset{\sim}{u}\cdot\underset{\sim}{u}}\underset{\sim}{u} = \frac{-7\cdot1+4\cdot2+1\cdot1}{1\cdot1+2\cdot2+1\cdot1}(1,2,1)$$

$$= \frac{2}{6}(1,2,1)$$

$$= \left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

(If you don't remember, then you can derive it:



$\text{proj}_{\underset{\sim}{u}}(\underset{\sim}{v})$ has length $\|\underset{\sim}{v}\|\cos\theta$ and direction vector $\underset{\sim}{u}$. If we normalize $\underset{\sim}{u}$ by taking $\dfrac{\underset{\sim}{u}}{\|\underset{\sim}{u}\|}$, then this gives a unit vector in the direction of $\underset{\sim}{u}$. Hence $\text{proj}_{\underset{\sim}{u}}(\underset{\sim}{v}) = \|\underset{\sim}{v}\|\cos\theta\dfrac{\underset{\sim}{u}}{\|\underset{\sim}{u}\|}$

$= \|\underset{\sim}{u}\|\cdot\|\underset{\sim}{v}\|\cos\theta\dfrac{\underset{\sim}{u}}{\|\underset{\sim}{u}\|^2} = \dfrac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\|\underset{\sim}{u}\|^2}\underset{\sim}{u}$.)

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5. Let $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (1, 1, 1)$ and $\mathbf{w} = (3, 1, 1)$ be vectors in $\mathbb{R}^3$. What is the volume of the parallelepiped with sides $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$?

A 2

B $\sqrt{2}$

C $2\sqrt{2}$

D 4

E 1

F $\frac{1}{\sqrt{2}}$

There is nothing to do except recall that this volume is given by $V = |(\underset{\sim}{u} \times \underset{\sim}{v}) \cdot \underset{\sim}{w}|$ (in any order).

$\underset{\sim}{u} \times \underset{\sim}{v} = (-1, 2, -1)$ and $(\underset{\sim}{u} \times \underset{\sim}{v}) \cdot \underset{\sim}{w} = -1 \cdot 3 + 2 \cdot 1 - 1 \cdot 1$
$= -2$

2 x 3 x 1 x 2
1 1 1 1

So $V = |-2| = 2$.

6. Let $P(-1, 1, 0)$, $Q(2, 0, -1)$, and $R(0, 1, 2)$ be points in $\mathbb{R}^3$. What is the area of triangle PQR?

A $\sqrt{54}$

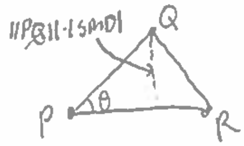
B $\frac{\sqrt{54}}{2}$

C $\sqrt{34}$

D $\frac{\sqrt{34}}{2}$

E $\frac{5}{2}$

F $\sqrt{26}$



(This triangle can be drawn in a number of ways.)

$A_\Delta = \frac{bh}{2} = \frac{1}{2} \|\underset{\sim}{PR}\| \cdot \|\underset{\sim}{PQ}\| \cdot |\sin\theta|$

$= \frac{1}{2} \|\underset{\sim}{PR} \times \underset{\sim}{PQ}\|$.

Well, $\underset{\sim}{PR} = (0-(-1), 1-1, 2-0) = (1, 0, 2)$
and $\underset{\sim}{PQ} = (2-(-1), 0-1, -1-0) = (3, -1, -1)$

Then $\underset{\sim}{PR} \times \underset{\sim}{PQ} = (2, 7, -1)$ and $\|\underset{\sim}{PR} \times \underset{\sim}{PQ}\| = \sqrt{2^2 + 7^2 + (-1)^2}$
$= \sqrt{54}$.

0 x 2 x 1 x 0
-1 -1 3 -1

So $A_\Delta = \frac{\sqrt{54}}{2}$

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7. Let π be the plane passing through the two points $(2,-1,4)$ and $(-1,3,0)$ such that π is perpendicular to the plane $3x-4y+5z=0$. Which of the following is an equation for π?

A $2x-y+4z=21$

B $3x-4z=4$

C $12x+4y-5z=0$

D $-x+3y=3$

E $4x+3y=5$

F $-x+3y=5$

We need the normal of π and a point on π. To find the normal, we need two direction vectors of π. Our first direction vector is obtained from the two given points: $\underset{\sim}{u}=(-1-2, 3-(-1), 0-4)=(-3,4,-4)$

Since π is perpendicular to a plane with normal vector $(3,-4,5)$ and the normal vector is already perpendicular to that plane, we can use $\underset{\sim}{v}=(3,-4,5)$ as the second direction vector for π.

Then the normal vector for π is

$\underset{\sim}{n}=\underset{\sim}{u}\times\underset{\sim}{v}=(20-16,-12+15,12-12)=(4,3,0)$

$\begin{matrix}4 & 4 & -3 & 4\\ -4 & 5 & 3 & -4\end{matrix}$

So an equation for π is $4x+3y+0z=d$. Sub in the point $(2,-1,4)$ to find d: $4(2)+3(-1)=5=d$.

So the equation for π that we want is $4x+3y=5$.

8. Which of the following is a point-normal form equation for the plane with vector parametric equation

$$\nu=(0,1,-1)+s(1,-1,2)+t(4,-6,2);\quad s,t\in\mathbb{R}$$

Use the two direction vectors to find the normal vector for ν:

A $y-3z=5$

B $x-y+2z=4$

C $2x+3y-8z=2$

D $3x-y-z=4$

E $3x-2y+2z=1$

F $10x+6y-2z=8$

$\underset{\sim}{n}=(1,-1,2)\times(4,-6,2)=(-2+12,8-2,-6+4)$

$=(10,6,-2)$

$\begin{matrix}-1 & 2 & 1 & -1\\ -6 & 2 & 4 & -6\end{matrix}$

Sub in the point $(0,1,-1)$ (given above) to $10x+6y-2z=d$ to get d:

$10(0)+6(1)-2(-1)=8=d$

Then the equation is $10x+6y-2z=8$.

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9. Let L be the line passing through the points $(-1,3,2)$ and $(1,2,1)$. What is the point of intersection of L with the plane $x+y-2z=4$?

- **A** $(3,1,0)$
- **B** $(2,1,0)$
- **C** $(2,1,1)$
- **D** $(3,1,1)$
- **E** $(3,1,2)$
- **F** $(3,2,2)$

First, write parametric equations for L. Use the point $(-1,3,2)$ and direction vector $(1-(-1), 2-3, 1-2) = (2,-1,-1)$ to write

$$\begin{cases} x = -1+2t \\ y = 3-t \\ z = 2-t \\ t \in \mathbb{R} \end{cases}$$

Then, sub these into the equation of the plane to find the value of t where the line and plane meet.

$$x+y-2z=4 \Rightarrow (-1+2t)+(3-t)-2(2-t)=4$$
$$\Leftrightarrow -2+3t=4$$
$$\Leftrightarrow t=\tfrac{6}{3}=2.$$

When $t=2$, the line is at the point $(-1+2\cdot2, 3-2, 2-2) = (3,1,$

10. Which of the following set of vectors in $\mathbb{R}^3$ is perpendicular to vectors $(-1,1,2)$ and $(1,2,3)$?

- **A** $\{(2,9,3)\}$
- **B** $\{(-1+t, 3+2t, t) | t \in \mathbb{R}\}$
- **C** $\{(t,0,2t) | t \in \mathbb{R}\}$
- **D** $\{(-t,5t,-3t) | t \in \mathbb{R}\}$
- **E** $\{(0,0,0)\}$
- **F** $\{(3,2,-10\}$

Take the cross product to find a vector perpendicular to them both:

$$(-1,1,2)\times(1,2,3) = (3-4, 2+3, -2-1)$$

$$\begin{matrix} 1 & & 2 & & -1 & & 1 \\ & \times & & \times & & \times & \\ 2 & & 3 & & 1 & & 2 \end{matrix} \qquad = (-1,5,-3)$$

Then the set of all vectors parallel to $(-1,5,-3)$ is the set of all scalar multiples of $(-1,5,-3)$. We can write this set as $\{t(-1,5,-3) \mid t\in\mathbb{R}\} = \{(-t,5t,-3t) \mid t\in\mathbb{R}\}$.