

# MAT 1341 Partial Exam I

March 4, 2019      Length: 80 minutes.

Professor:

Family name: \_\_\_\_\_

First name: \_\_\_\_\_

Student number: \_\_\_\_\_

1	D
2	F
3	F
4	
5	
6	
Total	

## PLEASE CAREFULLY READ THESE INSTRUCTIONS::

1. You have 80 minutes to write this exam.
2. You are not allowed to consult your notes or any books. Calculators, phones, and other electronic devices are not allowed.
3. Carefully read each question and **answer all questions in the space provided for this purpose.** For questions 4 to 6, you can use the back of the pages if necessary, but do not forget to indicate it to the T.A!
4. Questions 1 to 3 are multiple choice questions, each worth 1 point. No partial credit will be awarded. You must indicate the method you used to select the correct answer; unjustified answers will not be given credit.
5. Questions 4 to 6 are to be developed. They are each worth 6 points. **You must justify and write your answers correctly to get all possible points.**
6. Good luck! Bonne chance!



2. Which of the following sets are bases of  $\mathbb{R}^4$ ?

(1)  $\{(1, 0, 0, 0), (4, 2, 0, 0), (1, 4, 0, 0), (5, 6, 1, 0)\}$

(2)  $\{(0, -1, 2, 3), (0, 3, 3, 2), (4, 2, 0, 0)\}$

(3)  $\{(-1, 3, -5, 1), (1, -2, 4, 2), (2, 0, 4, 3), (5, 1, 9, 4), (4, 2, 0, 0)\}$

- A. All three are bases.  
 B. Only (1) is a basis.  
 C. Only (2) is a basis.  
 D. (1) and (2) are both bases.  
 E. (1) and (3) are both bases.  
 F. None of these three are bases of  $\mathbb{R}^4$

Solution: We know that  $\dim(\mathbb{R}^4) = 4$ .

option (2) is too small. option (3) is too big.  
 We need to check option (1) if it is L.I. We can do it by solving  $[A|0]$ .

$$\left[ \begin{array}{cccc|c} 1 & 4 & 1 & 5 & 0 \\ 0 & 2 & 4 & 6 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \sim \left[ \begin{array}{cccc|c} 1 & 4 & 1 & 5 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{rank}(A) = \text{rank}([A|0]) = 3$$

In addition,  $\text{rank}(A) = 3 < \# \text{ of columns} = 4$

Hence, this system  $[A|0]$  has infinitely many solutions  $\Rightarrow$  option (1) is L.D. consequently

option (1) cannot be a basis of  $\mathbb{R}^4$

Answer: F

3. Find the value of  $s$  for which  $(1, -4, s)$  belongs to  $\text{Span}\{(1, 2, 1), (2, -2, 6), (4, 14, 0)\}$

A.  $s = 0$

B.  $s = 1$

C.  $s = 2$

D.  $s = 3$

E.  $s = 4$

F.  $s = 5$

Solution: Set  $b = (1, -4, s) \in \text{Span}\{(1, 2, 1), (2, -2, 6), (4, 14, 0)\}$

$\iff [A|b]$  is consistent, where

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -2 & 14 \\ 1 & 6 & 0 \end{bmatrix}$$

We need to solve  $[A|b]$  (consistent)

$$[A|b] = \left[ \begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 2 & -2 & 14 & -4 \\ 1 & 6 & 0 & s \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 1 & -1 & 7 & -2 \\ 1 & 6 & 0 & s \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 0 & 3 & -3 & 3 \\ 0 & -4 & 4 & 1-s \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -4 & 4 & 1-s \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 5-s \end{array} \right] \text{ consistent}$$

$$\Rightarrow 5-s=0 \Rightarrow s=5$$

Answer:  F

4. Consider the vector space  $\mathbb{F}([-1, 1]) = \{f \mid f: [-1, 1] \rightarrow \mathbb{R}\}$  of functions, defined from  $[-1, 1]$  to  $\mathbb{R}$ , with the standard operations. Recall that the zero vector of  $\mathbb{F}([-1, 1])$  is the function that has the value 0 for all  $x \in [-1, 1]$ .

Define three functions  $f$ ,  $g$  and  $h$  de  $\mathbb{F}([-1, 1])$  by

$$f(x) = x, \quad g(x) = 1 - 2x + x^2, \quad \text{and} \quad h(x) = 1 + x^3.$$

Now let,  $W = \text{Span} \{f, g, h\}$

- Show that  $f$ ,  $g$  and  $h$  are linearly independent.
- Find the base of  $W$  and determine the dimension of  $W$ .
- If  $k \in \mathbb{F}([-1, 1])$  is the function defined by  $k(x) = 2 + x^2 + x^3$ , for  $x \in [-1, 1]$ , show that  $k \in W$ .
- What is the dimension of the subspace  $Y = \text{Span} \{f, g, h, k, \}$ ?  
(You must justify your answers)

Solution:  $W = \text{span} \{f, g, h\}$

$$\textcircled{a} \quad af(x) + bg(x) + ch(x) = 0 \Rightarrow a(x) + b(1 - 2x + x^2) + c(1 + x^3) = 0$$

$$\Rightarrow \underline{ax} + b - \underline{2bx} + \underline{bx^2} + c + \underline{cx^3} = 0$$

$$\Rightarrow cx^3 + bx^2 + (a - 2b)x + b + c = 0$$

$$\begin{cases} c = 0 \\ b = 0 \\ a - 2b = 0 \\ b + c = 0 \end{cases} \Rightarrow \begin{cases} c = 0 \\ b = 0 \\ a = 2b = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases}$$

we conclude that  $\{f, g, h\}$  is L.I

- Since  $\{f, g, h\}$  is L.I and  $W = \text{span} \{f, g, h\}$   
 $\Rightarrow \{f, g, h\}$  is a base of  $W$ .  $\dim(W) = 3$

© We have  $k(x) = 2 + x^2 + x^3 = (2x - 2x) + 2 + x^2 + x^3$

$$= 2x + 1 - 2x + x^2 + 1 + x^3$$

$$= 2(x) + (1 - 2x + x^2) + (1 + x^3)$$

$$= 2f(x) + 1g(x) + 1h(x)$$

$\Rightarrow k(x) \in \text{span}\{f, g, h\}$

④ Since  $k \in W = \text{span}\{f, g, h\}$ ,  $\text{span}\{f, g, h\} = \text{span}\{f, g, h, k\}$ . Therefore  $\dim(W) = \dim(Y) = 3$ .

5. Consider the following linear system:

$$\begin{aligned} x + y + 2z &= 1 \\ 2x + y + az &= 2 \\ 3x + 4y + az &= c \end{aligned}$$

- a) Calculate the rank of the coefficient matrix  $A$  and the rank of the augmented matrix  $[A|b]$ , in function of the parameters  $a$  and  $c$ .
- b) Deduce the values of  $a$  and  $c$  for which the system has
- (i) no solutions
  - (ii) infinitely many solutions
  - (iii) unique solution.
- c) In case b (ii) above, find the general solution of the system and give a complete geometrical description of the set of solutions.

Solution (a) We need to solve the linear system

$$[A|b] = \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & 1 & a & 2 \\ 3 & 4 & a & c \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & a-4 & 0 \\ 0 & 1 & a-6 & c-3 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -a+4 & 0 \\ 0 & 1 & a-6 & c-3 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -a+4 & 0 \\ 0 & 0 & -2a+10 & 3-c \end{array} \right]$$

$$\text{rank}(A) = \begin{cases} 2 & \text{if } a=5 \\ 3 & \text{if } a \neq 5 \end{cases}$$

$$\text{rank}([A|b]) = \begin{cases} 2 & \text{if } a=5 \text{ and } c=3 \\ 3, & \text{otherwise.} \end{cases}$$

8 (b) (i) No solutions:  $\Rightarrow \text{rank}(A) < \text{rank}([A|b])$

$$\Leftrightarrow a=5 \text{ and } c \neq 3$$

(ii) Infinitely many solutions

$$\text{rank}(A) = \text{rank}([A|b])$$

and

$$\text{rank}(A) < \# \text{ of columns}$$

$$\Leftrightarrow a=5 \text{ and } c=3$$

(iii) a unique solution

$$\text{rank}(A) = \text{rank}([A|b])$$

and

$$\text{rank}(A) = \# \text{ of columns}$$

$$\Leftrightarrow a \neq 5 \text{ and for all } c$$

(3) In case (ii) Infinitely many solutions:  $(a=5, c=3)$

$$[A|b] = \left[ \begin{array}{ccc|c} \textcircled{1} & 1 & 2 & 1 \\ 0 & \textcircled{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} \textcircled{1} & 0 & 3 & 1 \\ 0 & \textcircled{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$x_1$  and  $x_2$  are leading variables  
 $x_3$  is non-leading variable ( $x_3 = s$ )

$$\begin{aligned} x_1 &= -3s + 1 \\ x_2 &= s \\ x_3 &= s \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3s+1 \\ s \\ s \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Geometric description  
 this is a

line in  $\mathbb{R}^3$   
 through  $(1, 0, 0)$  with  
 direction  
 vector  $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

6. In each case, indicate in the corresponding box if the statement below is (always) true or can be false.

- If you feel that the statement may be wrong, give an explicit example that it is false.
- If you feel that the statement is (always) true, you must justify it with a clear explanation.

I. A system of 10 linear equations with 15 unknowns is consistent with many infinitely solutions.

15 variables

10 equations

$$\left[ \begin{array}{cccc|c} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ \hline 0 & \dots & 0 & & 1 \end{array} \right]$$

ANSWER

False

Last row: degenerate equation  $\Rightarrow$  inconsistent system

II. If the coefficient matrix  $A$  of a homogeneous linear system of 8 equations with 7 unknowns such that  $\text{rank}(A) = 7$ , then the system has a unique solution.

7 variables

8 equations

$$\left[ \begin{array}{cccc|c} & & & & 0 \\ & & & & 0 \\ & & & & \vdots \\ & & & & 0 \end{array} \right]$$

ANSWER

True

\* homogeneous system is always consistent ( $\text{rank}(A) = \text{rank}([A|0])$ )

\* Since  $\text{rank}(A) = \# \text{ of variables} = 7 \Rightarrow$  the system has a unique solution.

III. If the coefficient matrix of a consistent linear system has a column of zeros, then the system has many infinitely solutions.

\* This linear system is consistent

$$\Rightarrow \text{rank}(A) = \text{rank}([A|b]) \quad (1)$$

ANSWER

True

\* This linear system has a column of zeros

$$\Rightarrow \text{rank}(A) < \# \text{ of columns} \quad (2)$$

From (1) and (2), we conclude that this linear system has many infinitely solutions

IV. Consider two vectors  $u$  and  $v$  from a vector space  $V$ .

a) Give the definition of " $\{u, v\}$  is an <sup>linearly</sup> independent set."

$$\{u, v\} \text{ is l.I iff } au + bv = 0 \Rightarrow a = b = 0$$

b) If  $\{u, v\}$  is a base of  $\mathbb{R}^2$ , then  $\{u - v, v\}$  is linearly independent.

Method 1

ANSWER

True

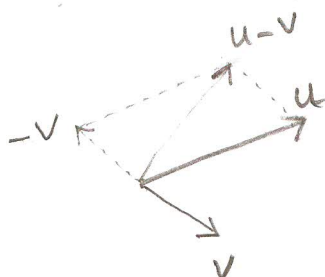
$\{u, v\}$  is a base of  $\mathbb{R}^2 \Rightarrow \{u, v\}$  is L.I

$\Rightarrow u$  and  $v$  are not collinear. ( $u \neq kv$ ) or

$v \neq ku$  for some  $k \in \mathbb{R}$ . For example



Now, to prove that  $\{u - v, v\}$  is L.I, we can just prove that  $u - v$  and  $v$  are not collinear



$u - v$  and  $v$  are not collinear  $\Rightarrow \{u - v, v\}$  is L.I

Method 2

$$\{u, v\} \text{ is L.I} \Rightarrow au + bv = 0 \Rightarrow a = b = 0$$

$$\Rightarrow au - av + bv = 0$$

$$\Rightarrow a(u - v) + bv = 0 \Rightarrow a = b = 0$$

Hence,  $\{u - v, v\}$  is L.I