UNIVERSITY OF
WATERLOO

Department of Economics
ECON 391-001
Assignment 1 - Answers

L-A. Busch **September 11, 2019**

Question [1]:

1. (Cobb Douglas)

   (a) Hard to draw properly but they are 'bowed in' to the origin and are 'radial blowups' of each other, that is, all indifference curves have the same slope along a ray from the origin. [Just compute the MRS!] It is worthwhile to note that indifference curves are asymptotic to the axes but never touch the axes. Indeed, the coordinate axes are the "zeroth" indifference curve – which therefore has a Leontief shape! This then also means that Cobb-Douglas is not strictly monotonic, since if I have nothing of good 1, any additional amounts of good 2 yield no utility.

   (b) $L(x_1, x_2, \lambda) = x_1^{\alpha} x_2^{1-\alpha} - \lambda\,(p_1 x_1 + p_2 x_2 - m)$

   (c)
$$\begin{aligned} \frac{\partial L(\cdot)}{\partial x_1} &= \alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 &&= 0 \\ \frac{\partial L(\cdot)}{\partial x_2} &= (1-\alpha) x_1^{\alpha} x_2^{-\alpha} - \lambda p_2 &&= 0 \\ \frac{\partial L(\cdot)}{\partial \lambda} &= p_1 x_1 + p_2 x_2 &&= w \end{aligned}$$

   (d) There are two options: isolate λ from the first equation, use that expression to eliminate λ in the second. Then solve the resulting equation for one of the x_i and use that expression to eliminate x_i in the third (the budget). You can now solve the budget for x_j, and then substitute back into the x_i equation. Alternatively, you can bring λp_i to the right hand side in both of the first two equations and then take the ratio of them. That eliminates λ, and you can now solve the resulting equation for one of the x_i and use that in the budget.

   (e)
$$x(p, w) = \begin{bmatrix} \frac{\alpha w}{p_1} \\ \frac{(1-\alpha) w}{p_2} \end{bmatrix}$$

2. (general Leontief)

   (a) Indifference curves are L shaped, increasing to the up and right. The 'kinks' all lie along the line $x_2 = a x_1$. To see this you may want to note that this utility function could also be written as follows:
$$u(x) = \begin{cases} a x_1 & \text{if } a x_1 \leq x_2 \\ x_2 & \text{if } a x_1 \geq x_2 \end{cases}$$

This makes it clear that indifference curves are either vertical (depend only on x_1) or horizontal (depend only on x_2) and that there is a switch between these two cases.

(b) $L(x_1, x_2, \lambda) = \min\{ax_1, x_2\} - \lambda(p_1x_1 + p_2x_2 - m)$

(c) We cannot compute first order conditions since the function is not differentiable everywhere. However, by inspection (of the diagram - just superimpose an arbitrary budget on what was drawn in a)) we can see that the optimal solution must occur at the kink (where the function switches). That gives us our first equation. The budget is the second.

$$\begin{aligned} ax_1 &= x_2 \\ p_1x_1 + p_2x_2 &= w \end{aligned}$$

(d) Solve the system: this should be obvious.

(e)

$$x(p, w) = \begin{bmatrix} \frac{w}{p_1+ap_2} \\ \frac{aw}{p_1+ap_2} \end{bmatrix}$$

3. These preferences are a form of quasi-linear preferences (with good 1 being the numeraire commodity). [Aside: quasi-linear refers to any preference the is linear in one good, so in general they have the form $u(x) = x_i + v(x_{-i})$, where x_{-i} refers to all goods that are not good i, and $v(\cdot)$ is some regular quasi-concave function.] The non-linear part is clearly concave (i.e. has diminishing marginal utility) so that we know that the function overall is nice and quasi-concave (which means that preferences are nice and convex.) Indifference curves are horizontal transposes - simply shifted along the horizontal axis. They cut the vertical axis, so there is the possibility of a corner solution. In contrast to Cobb-Douglas, they terminate on the horizontal axis, but at a zero slope (meaning no corner solutions there at strictly positive prices.) So we either need to use Kuhn-Tucker conditions instead of a simple Lagrangian, or we ignore that problem initially and deal with it later. The Lagrangian is

$$L(p, w, \lambda) = x_1 + 10 \ln x_2 - \lambda(p_1x_1 + p_2x_2 - w)$$

The first order conditions are

$$\begin{aligned} \frac{\partial L(\cdot)}{\partial x_1} &= 1 - \lambda p_1 &&= 0 \\ \frac{\partial L(\cdot)}{\partial x_2} &= 10/x_2 - \lambda p_2 &&= 0 \\ \frac{\partial L(\cdot)}{\lambda} &= p_1x_1 + p_2x_2 &&= w \end{aligned}$$

The first two of those indicate that $x_2 = 10p_1/p_2$. Using the budget we obtain that $x_1 = w/p_1 - 10$ and we note that this may be negative. Since it is not allowed to be negative, that would imply that it is zero. In that case the tangency condition does not apply (you can verify that the indifference curve is flatter than the budget at such points), and we simply solve the budget (at $x_1 = 0$) to obtain $x_2 = w/p_2$. So Walrasian Demand is

$$x(p,w) = \begin{cases} \begin{bmatrix} w/p_1 - 10 \\ 10p_1/p_2 \end{bmatrix} & \text{if } w > 10p_1 \\ \begin{bmatrix} 0 \\ w/p_2 \end{bmatrix} & \text{otherwise} \end{cases}$$

4. (perfect substitute) Indifference curves are parallel linear lines (with slope $-a$.) We cannot solve them using calculus techniques and a Lagrangian since we either have no answer (budget coincides with an indifference curve) or a corner solution. This is evidenced by the Lagrangian "not working":

$$L(p,w,\lambda) = ax_1 + x_2 - \lambda(p_1x_1 + p_2x_2 - w)$$

The first order conditions are

$$\begin{aligned} \frac{\partial L(\cdot)}{\partial x_1} &= a - \lambda p_1 &&= 0 \\ \frac{\partial L(\cdot)}{\partial x_2} &= 1 - \lambda p_2 &&= 0 \\ \frac{\partial L(\cdot)}{\lambda} &= p_1x_1 + p_2x_2 &&= w \end{aligned}$$

Note that the first two equations either are or are not true jointly.

We argue the corner solutions from "inspection" (instead of using more formal 'complementary slackness' condition arguments.) There are 2 pure corner solutions, where the consumer consumes only one of the commodities. So we get

$$x(p,w) = \begin{bmatrix} w/p_1 \\ 0 \end{bmatrix} \text{ if } a \geq \frac{p_1}{p_2}$$

$$x(p,w) = \begin{bmatrix} 0 \\ w/p_2 \end{bmatrix} \text{ if } a \leq \frac{p_1}{p_2}$$

In addition there exists the set of choices on the budget if $a = p_1/p_2$ and the budget coincides with an indifference curve. So technically this is not a function at all but a correspondence (a relationship that associates a set of outputs with a given input. Another interesting aside is the fact that the set of choices is always a convex set. This is caused by convexity of preferences and is not true, for example, in question 6 below!)

5. This one is an interesting little function. It is an example of a function that does not have linear income expansion paths (see next assignment) as all the previous examples had! Indifference curves are L shaped as for a Leontief function, however, the kink occurs where $x_1 = \sqrt{x_2}$. Demands are easily solved for. While we cannot use a differentiable approach here, it is clear that optimal consumption occurs at the kink. So we again solve the "kink equation" and the budget simultaneously. We get (using the quadratic formula to solve for $\sqrt{x_2}$) that

$$x(p,w) = \begin{bmatrix} \frac{\sqrt{p_1^2 + 4p_2w} - p_1}{2p_2} \\ \frac{(\sqrt{p_1^2 + 4p_2w} - p_1)^2}{4p_2^2} \end{bmatrix}$$

6. Indifference curves here are upside down L shapes - they start horizontally at the vertical axis and then kink to slope vertically down until they intersect the horizontal axis. You may more easily see this from the alternate way to write the function:

$$u(x) = \begin{cases} ax_1 & \text{if } ax_1 \geq x_2 \\ x_2 & \text{if } ax_1 \leq x_2 \end{cases}$$

Note that these are NOT convex preferences. So we are not looking for a tangency or its 'equivalent' - the place where the kink is on the budget. Instead, if you draw a budget you will see that we have corner solutions, just like for perfect substitutes - but without the multi-valued part along the budget line. Geometry tells us that the slope of the line that connects the two endpoints of an indifference curve is $-a$ — just as for the perfect substitute prefernces we had before. So demand is

$$x(p, w) = \begin{bmatrix} w/p_1 \\ 0 \end{bmatrix} \text{ if } a \geq \frac{p_1}{p_2}$$

$$x(p, w) = \begin{bmatrix} 0 \\ w/p_2 \end{bmatrix} \text{ if } a \leq \frac{p_1}{p_2}$$

But in this case there is no set along the budget. Indeed, if you draw a demand curve (next assignment!) it has a hole.

Question [2]: The Lagrangian is

$$L(p, w, \lambda) = x_1^{\alpha} x_2 \beta x_3^{1-\alpha-\beta} - \lambda(p_1x_1 + p_2x_2 + p_3x_3 - w)$$

The first order conditions for this are

$$\begin{aligned} \frac{\partial L(\cdot)}{\partial x_1} &= \alpha u(x)/x_1 - \lambda p_1 &= 0 \\ \frac{\partial L(\cdot)}{\partial x_2} &= \beta u(x)/x_2 - \lambda p_2 &= 0 \\ \frac{\partial L(\cdot)}{\partial x_3} &= (1 - \alpha - \beta)u(x)/x_3 - \lambda p_3 &= 0 \\ \frac{\partial L(\cdot)}{\lambda} &= p_1x_1 + p_2x_2 + p_3x_3 &= w \end{aligned}$$

(Here I save on typing since the derivative of $v(x) = x^a$ is $ax^{(a-1)} = ax^a/x = av(x)/x$.) I am NOT providing the full algebra here, solving this system is just an application of standard highschool/ECON 211 algebra techniques. You ought to be able to arrive at the answer yourself. Note the general principle of solving systems of equations: you use one equation to solve for one variable in terms of all the others. You use that expression in all remaining equations to eliminate that variable. You now have a smaller system of equations (one less, since you cannot reuse the equation you already used) and one variable less. Repeat. Keep repeating until you have one equation in one variable. Solve that, and then use that value to get the second last varible, both of those for the third last, and so on.

$$x(p, w) = \begin{bmatrix} \frac{\alpha w}{p_1} \\ \frac{\beta w}{p_2} \\ \frac{(1-\alpha-\beta)w}{p_3} \end{bmatrix}$$

Question [3]: This does not take long since we should realize that we have already solved the general case, so now we simply substitute the assumed demand parameter of the utility function in the general solution, and we use the fact that $w = 10p_1 + 5p_2$. So we get

1.
$$x(p,w) = \begin{bmatrix} \frac{4(10p_1+5p_2)}{10p_1} \\ \frac{6(10p_1+5p_2)}{10p_2} \end{bmatrix}$$

2.
$$x(p,w) = \begin{bmatrix} 5p_2/p_1 \\ 10p_1/p_2 \end{bmatrix}$$

3.
$$x(p,w) = \begin{bmatrix} 10 + 5p_2/p_1 \\ 0 \end{bmatrix} \text{ if } 3 \geq \frac{p_1}{p_2}$$
$$x(p,w) = \begin{bmatrix} 0 \\ 5 + 10p_1/p_2 \end{bmatrix} \text{ if } 3 \leq \frac{p_1}{p_2}$$

4.
$$x(p,w) = \begin{bmatrix} \frac{10p_1+5p_2}{p_1+2p_2} \\ \frac{20p_1+10p_2}{p_1+2p_2} \end{bmatrix}$$