[42] **1. Short-Answer Questions.** Put your answer in the box provided but show your work also. Each question is worth 3 marks, but not all questions are of equal difficulty.

(a) Find an equation of the plane parallel to the plane $x - 3y + 2z = 1$ passing through the point $(1, 0, -1)$.

Answer:

(b) Are the level curves of the paraboloid $z = x^2 + y^2$ lines, circles, parabolas, hyperbolas or ellipses?

Answer:

(c) Let $f(x, y) = y^3 \cos(2x)$. Find $\dfrac{\partial^2 f}{\partial x \partial y}$.

Answer:

(d) $\sum_{k=1}^{4} f(1+k) \cdot 1$ is a left Riemann sum for a function $f(x)$ on the interval $[a, b]$ with n sub intervals. Find the values of a, b and n.

Answer:
a= , b= , n=

(e) Suppose $\int_2^3 f(x)\,dx = -1$ and $\int_2^3 g(x)\,dx = 5$. Evaluate $\int_2^3 \big(6f(x) - 3g(x)\big)\,dx$.

Answer:

(f) For the function $f(x) = x^3 - \sin 2x$, find its antiderivative $F(x)$ that satisfies $F(0) = 1$.

Answer:

(g) Evaluate $\dfrac{d}{dx}\left(\displaystyle\int_0^{\sin x}(t^6+8)\,dt\right)$.

Answer:

(h) Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2+25}}$.

Answer:

(i) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} x\cos x\,dx$.

Answer:

(j) Evaluate $\int \cos^3 x \, dx$.

Answer:

(k) Evaluate the integral $\int_0^1 \frac{x^4}{x^5 - 1} \, dx$ or state that it diverges.

Answer:

(l) Evaluate $\displaystyle\sum_{k=7}^{\infty} \frac{1}{8^k}$.

Answer:

(m) Solve the differential equation $y'(t) = e^{\frac{y}{3}} \cos t$. You should express the solution $y(t)$ in terms of t explicitly.

Answer:

(n) Find the limit of the sequence $\left\{ \ln\left(\sin\frac{1}{n}\right) + \ln(2n) \right\}$.

Answer:

Full-Solution Problems. In questions 2 – 6, justify your answers and show all your work.

[12] **2.** (a) Evaluate $\int \frac{e^x}{(e^x+1)(e^x-3)}\,dx$.

2.(b) Evaluate $\int_2^4 \frac{x^2 - 4x + 4}{\sqrt{12 + 4x - x^2}}\, dx$.

[12] **3.** Let $f(x, y) = (x-1)^2 + (y+1)^2$.

(a) Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x, y)$ on the circle $x^2 + y^2 = 4$. A solution that does not use the method of Lagrange multipliers will receive no credit, even if the answer is correct.

3.(b) Find the maximum and minimum values of the function $f(x, y)$ over the region $R = \{(x, y) : x^2 + y^2 \leq 4\}$.

[10] 4. A continuous random variable X is given by the following probability density function

$$f(x) = \begin{cases} \frac{1}{4} + \frac{1}{2}|x| & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the expected value $E(X)$ of the random variable X.

(b) Let $F(x)$ be the cumulative distribution function for the random variable X. Find $F(x)$ for $0 < x < 1$.

[16] **5.**

(a) Suppose that $\dfrac{df}{dx} = \dfrac{x}{1+3x^3}$ and $f(0) = 1$. Find the Maclaurin series for $f(x)$.

(b) Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+n+1}{n^5-n}$ converges or diverges.

5.(c) Determine whether the series $\sum_{m=1}^{\infty} \frac{3m + \sin\sqrt{m}}{m^2}$ converges or diverges.

5.(d) Determine whether the series $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$ converges or diverges.

[8] **6**. Suppose that the series $\sum_{n=0}^{\infty}(1 - a_n)$ converges, where $a_n > 0$ for $n = 0, 1, 2, 3, \cdots$.

(a) Determine whether the series $\sum_{n=0}^{\infty} 2^n a_n$ converges or diverges.

(b) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$.

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1. (a) $1\cdot(x-1)-3(y-0)+2(z+1)=0$ or $x-3y+2z=-1$.

(b) circles.

(c) $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial x}(y^3\cos 2x)\right) = \dfrac{\partial}{\partial y}\left(y^3(-2)\sin 2x\right) = 3y^2(-2)\sin 2x$

$= -6y^2 \sin 2x$.

(d) $a=2$, $b=6$, $n=4$.

(e) $\int_2^3 (6f-3g)\,dx = 6\int_2^3 f\,dx - 3\int_2^3 g(x)$

$= 6(-1)-3\cdot 5 = -21$

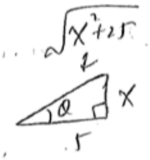
(f) $F(x) = \int f(x)\,dx = \int (x^3 - \sin 2x)\,dx = \dfrac{x^4}{4} + \dfrac{1}{2}\cos 2x + c$

$1 = F(0) = \dfrac{1}{2}\cos 0 + c = \dfrac{1}{2} + c \Rightarrow c = \dfrac{1}{2}$. Hence,

$F(x) = \dfrac{x^4}{4} + \dfrac{1}{2}\cos 2x + \dfrac{1}{2}$

(g) $(\sin^6 x + 8)\cos x$.

(h) $\displaystyle\int \frac{dx}{\sqrt{x^2+25}} \underset{x=5\tan\theta}{=} \int \frac{5\sec^2\theta\, d\theta}{\sqrt{(5\tan\theta)^2+25}} = \int \sec\theta\, d\theta$



$= \ln|\sec\theta + \tan\theta| + C = \ln\left|\dfrac{\sqrt{x^2+25}}{5} + \dfrac{x}{5}\right| + C$.

(i) $\displaystyle\int_0^{\pi/2} x\cos x\,dx \underset{\substack{u=x,\ v'=\cos x\\ u'=1,\ v=\sin x}}{=} (x\sin x)\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin x\,dx$

$= \dfrac{\pi}{2} + \cos x\Big|_0^{\pi/2} = \dfrac{\pi}{2} + \left(\cos\dfrac{\pi}{2} - \cos 0\right) = \dfrac{\pi}{2} - 1$.

(j) $\displaystyle\int \cos^3 x\,dx = \int \cos x \cos^2 x\,dx = \int \cos x(1-\sin^2 x)\,dx \underset{\substack{u=\sin x\\ du=\cos x\,dx}}{=} \int (1-u^2)\,du$

$= u - \dfrac{u^3}{3} + C = \sin x - \dfrac{1}{3}\sin^3 x + C$.

5-D

1. (k) $\int_0^1 \frac{x^4}{x^5-1}dx = \lim_{t\to 1^-} \int_0^t \frac{x^4}{x^5-1}dx$

$= \lim_{t\to 1^-} \left(\frac{1}{5}\ln|x^5-1|\right)\Big|_0^t = \lim_{t\to 1^-} \frac{1}{5}\ln|t^5-1| = -\infty,$

so the integral is divergent.

(l) $\left(\frac{1}{87}\right)/\left(1-\frac{1}{8}\right).$

(m) $\frac{dy}{dt} = e^{y/3}\cos t \Rightarrow e^{-y/3}dy = \cos t\, dt$

$\int e^{-y/3}dy = \int \cos t\, dt \Rightarrow -3e^{-y/3} = \sin t + C$

$e^{-y/3} = \ln\left(-\frac{1}{3}\sin t - \frac{C}{3}\right) \Rightarrow y = -3\ln\left(-\frac{1}{3}\sin t - \frac{C}{3}\right).$

(n) $\lim\left(\ln\left(\sin\frac{1}{n}\right) + \ln 2n\right) = \lim \ln\left(2n\sin\frac{1}{n}\right)$

$= \lim \ln\left(2\cdot\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right) = \ln 2.$

2. (a) $\displaystyle\int \frac{e^x}{(e^x+1)(e^x-3)}\,dx \underset{\substack{u=e^x \\ du=e^x dx}}{=\!=} \int \frac{du}{(u+1)(u-3)}.$

$$\frac{1}{(u+1)(u-3)} = \frac{A}{u+1} + \frac{B}{u-3} \Rightarrow 1 = A(u-3) + B(u+1)$$

$u=-1 \Rightarrow \quad 1 = A(-4) \Rightarrow A = -1/4.$

$u=3 \Rightarrow \quad 1 = B(4) \Rightarrow B = 1/4.$

Hence, $\displaystyle\int \frac{e^x}{(e^x+1)(e^x-3)}\,dx = \int \left(\frac{-1/4}{u+1} + \frac{1/4}{u-3}\right) du$

$$= -\frac{1}{4}\ln|u+1| + \frac{1}{4}\ln|u-3| + C = -\frac{1}{4}\ln|e^x+1| + \frac{1}{4}\ln|e^x-3| + C.$$

(b) $\displaystyle\int_2^4 \frac{x^2-4x+4}{\sqrt{12+4x-x^2}}\,dx = \int_2^4 \frac{x^2-4x+4}{\sqrt{12-(x^2-4x)}}\,dx = \int_2^4 \frac{x^2-4x+4}{\sqrt{12-(x^2-4x+4-4)}}\,dx$

$$= \int_2^4 \frac{(x-2)^2}{\sqrt{16-(x-2)^2}}\,dx \underset{\substack{x-2=4\sin\theta \\ dx=4\cos\theta}}{=\!=} \int_0^{\pi/6} \frac{(4\sin\theta)^2}{\sqrt{16-(4\sin\theta)^2}} \cdot 4\cos\theta\, d\theta$$

$$= 4^2 \int_0^{\pi/6} \sin^2\theta\, d\theta = 16\int_0^{\pi/6} \frac{1-\cos 2\theta}{2}\, d\theta = 8\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/6} = 8\left(\frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right).$$

3. (a) Let $g(x,y) = x^2+y^2-4$. Then we need to solve the system $\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{cases}$ or $\begin{cases} 2(x-1) = \lambda(2x) & (1) \\ 2(y+1) = \lambda(2y) & (2) \\ x^2+y^2 = 4 & (3) \end{cases}$

(1) $y \Rightarrow$ $xy - y = \lambda xy$ (4)

(2) $x \Rightarrow$ $xy + x = \lambda xy$ (5)

(4) and (5) $\Rightarrow$ $y = -x$. (6)

By (3) and (6), we get $x^2 + (-x)^2 = 4 \Rightarrow x = \pm\sqrt{2} \Rightarrow y = \mp\sqrt{2}$.

$f(\sqrt{2}, -\sqrt{2}) = (\sqrt{2}-1)^2 + (-\sqrt{2}+1)^2 = 2(\sqrt{2}-1)^2 = 2(2-2\sqrt{2}+1) = 6-4\sqrt{2}$.

$f(-\sqrt{2}, \sqrt{2}) = (-\sqrt{2}-1)^2 + (\sqrt{2}+1)^2 = 2(\sqrt{2}+1)^2 = 2(2+2\sqrt{2}+1) = 6+4\sqrt{2}$.

Hence, the maximum value of $f(x,y)$ on the circle is $6+4\sqrt{2}$,
the minimum value of $f(x,y)$ on the circle is $6-4\sqrt{2}$.

(b) $\begin{cases} \dfrac{\partial f}{\partial x} = 2(x-1) = 0 \\ \dfrac{\partial f}{\partial y} = 2(y+1) = 0 \end{cases} \Rightarrow$ $(1,-1)$ is the only critical point. $f(1,-1) = (1-1)^2 + (-1+1)^2 = 0$. (7)

By the conclusions in (a) and (7), we know that
the maximum value of $f(x,y)$ on the region R is $6+4\sqrt{2}$,
the minimum value of $f(x,y)$ on the region R is 0.

4. (a) $E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_{-1}^{1} x \left(\frac{1}{4} + \frac{1}{2}|x|\right) dx$

$= \int_{-1}^{0} x\left(\frac{1}{4} - \frac{1}{2}x\right) dx + \int_{0}^{1} x\left(\frac{1}{4} + \frac{1}{2}x\right) dx$

$= \int_{-1}^{0} \left(\frac{1}{4}x - \frac{1}{2}x^2\right) dx + \int_{0}^{1} \left(\frac{1}{4}x + \frac{1}{2}x^2\right) dx$

$= \left(\frac{1}{4}\frac{x^2}{2} - \frac{1}{2}\cdot\frac{1}{3}x^3\right)\Big|_{-1}^{0} + \left(\frac{1}{4}\frac{x^2}{2} + \frac{1}{2}\cdot\frac{1}{3}x^3\right)\Big|_{0}^{1}$

$= -\left(\frac{1}{8} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{6}\right) = 0.$

(b) $F(x) = \int_{-\infty}^{x} f(t) dt = \int_{-\infty}^{-1} f(t)dt + \int_{-1}^{0} f(t) dt + \int_{0}^{x} f(t) dt$

$= 0 + \int_{-1}^{0} \left(\frac{1}{4} - \frac{1}{2}t\right) dt + \int_{0}^{x} \left(\frac{1}{4} + \frac{1}{2}t\right) dt$

$= \left(\frac{1}{4}t - \frac{1}{2}\frac{t^2}{2}\right)\Big|_{-1}^{0} + \left(\frac{1}{4}t + \frac{1}{2}\frac{t^2}{2}\right)\Big|_{0}^{x}$

$= -\left(-\frac{1}{4} - \frac{1}{4}\right) + \frac{1}{4}x + \frac{1}{4}x^2 = \frac{1}{2} + \frac{1}{4}x + \frac{1}{4}x^2.$

5. (a) $f'(x) = x \cdot \frac{1}{1-(-3x^3)} = x \sum_{n=0}^{\infty} (-3x^3)^n = \sum_{n=0}^{\infty} (-1)^n 3^n \cdot x^{3n+1}$

$f(x) = \int \left(\sum_{n=0}^{\infty} (-1)^n 3^n x^{3n+1} \right) dx = C + \sum_{n=0}^{\infty} (-1)^n \cdot 3^n \cdot \frac{x^{3n+2}}{3n+2}$.

With $f(0)=1$, we have $C=1$ so $f(x) = 1 + \sum_{n=0}^{\infty} (-1)^n 3^n \cdot \frac{x^{3n+2}}{3n+2}$.

(b) $\lim_{n \to \infty} \frac{(n^2+n+1)/(n^5-n)}{(1/n^3)} = \lim \frac{n^5+n^4+n^3}{n^5-n} =$

$= \lim \frac{1+(1/n)+(1/n^2)}{1-\left(\frac{1}{n^4}\right)} = 1$. Since $\sum \frac{1}{n^3}$ converges,

$\sum_{n=0}^{\infty} \frac{n^2+n+1}{n^5-n}$ converges.

(c) Since $\frac{3m+\sin\sqrt{m}}{m^2} > \frac{3m-1}{m^2}$ and

$\sum \frac{3m-1}{m^2}$ diverges, $\sum \frac{3m+\sin\sqrt{m}}{m^2}$ diverges

(or $\frac{3m+\sin\sqrt{m}}{m^2} > \frac{3m-m}{m^2} = \frac{2m}{m^2} = \frac{2}{m}$)

(d) $f(x) = \frac{1}{x(\ln x)^3}$ is positive, decreasing and continuous for $x \geq 2$. Since

$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{t \to \infty} \int_2^t \frac{1}{x(\ln x)^3}\,dx$

$\underset{\substack{u=\ln x \\ du=\frac{dx}{x}}}{=} \lim_{t \to \infty} \int_{\ln 2}^{\ln t} \frac{du}{u^3} = \lim_{t \to \infty} \left(-\frac{1}{2u^2} \right) \Big|_{\ln 2}^{\ln t} = \frac{1}{2(\ln 2)^2}$,

the series $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$ converges.

6. (a) Since $\sum_{n=0}^{\infty} (1-a_n)$ converges, $\lim (1-a_n) = 0$ or $\lim a_n = 1$. Hence, $\lim (2^n a_n) = \infty$, which implies that $\sum 2^n a_n$ diverges.

(b) For $|x|>1$, $\lim |a_n x^n| = \lim (a_n |x|^n) = \infty$

Hence, $\lim a_n x^n \neq 0$. This proves that

$\sum a_n x^n$ diverges for $|x|>1$. (1)

Also, for $|x|<1$, with $x \neq 0$, we have

$$\lim \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim \left(\frac{a_{n+1}}{a_n} |x| \right) = \frac{\lim a_{n+1}}{\lim a_n} \cdot |x| = |x| < 1$$

$\Rightarrow$ $\sum a_n x^n$ converges for $|x|<1$ (2)

By (1) and (2), the radius of convergence is 1.