MAT1322, Final exam Practice sheet:

Integration

Area, Volumes, Applications of Integrals

Compute the area of the following regions:

1. Bounded region delimited by $y = e^x, y = 1, x = 2$.
2. Bounded region delimited by $y = \sin(x), y = \cos(x), x = 0, y = 0, x = \frac{\pi}{2}$.
3. Bounded region delimited by $y = 3x - 6, y = 9/x, x = 4$.

Compute the volume of the following solids:

1. $y = e^x, y = 1, x = 2$ rotated around $y = -2$.
2. $y = \sin(x), y = \cos(x), x = 0, y = 0, x = \frac{\sqrt{2}}{2}$ about $y = -1$.
3. $y = 3x - 6, y = 9/x, y = 0, x = 4$ about $x = -2$.

Applications of integrals:

1. A cable that weighs $2kg/m$ is used to lift $800kg$ of coal up to a mine shaft which is $500m$ deep. Find the work done.
2. Suppose that 2 J of work is needed to stretch a spring from its natural length of 30 cm to a length of 42 cm. How much work is needed to stretch the spring from 35 cm to $40cm$? Hint: you may use Hooke's law, F=kx to compute k from the numbers given and then apply this to the question.
3. A leaky 10 kg bucket is lifted from the ground to a height of 12 m at a constant speed with a rope that weighs 0.8 kg/m. Initially the bucket contains 36 kg of water, but the water leaks at a constant rate and finishes draining just as the bucket reaches the 12 m level. How much work is done?
4. A tank with a capacity of 500 liters contains 500 liters of water with 100 kg of salt in solution. Water containing 1 kg of salt per liter is entering at the rate of 3 liters per minute, and the mixture is allowed to flow out of the tank at a rate of 3 liters per minute. Find the amount of salt in the tank after 5 minutes.
5. A tank in the shape of an inverted cone has a height of 15 m and a base radius of 4 m and is filled with water to a depth of 12 meters. Determine the amount of work needed to pump all of the water to the top of the tank. Assume that the density of the water is $1000kg/m^3$.
6. A dam has the shape of the trapezoid. The height is 16 meters and the width is 92 meters at the top, and 60 meters at the bottom. The water level is 4 meters below the top of the dam. Let x be the depth of a horizontal stripe of the dam. Suppose ρ is the density of water, and g is the acceleration of gravity. Find the force acting on the dam (in Newtons).

Improper Integrals

Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

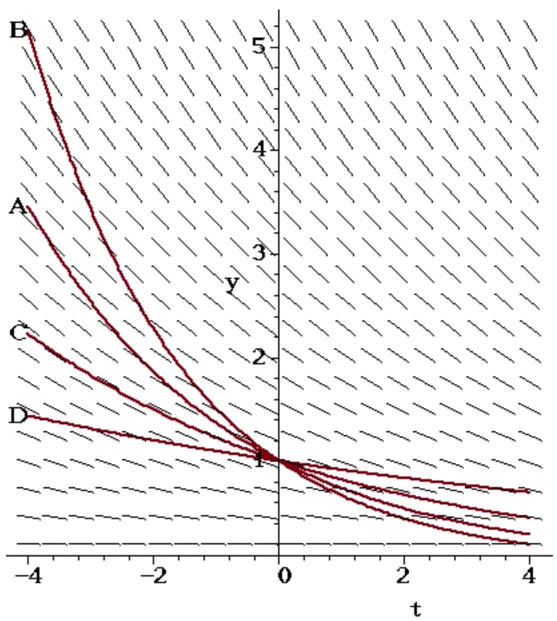
1. $\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx$.
2. $\int_1^5 \frac{1}{(x-2)^{2/3}}\,dx$.
3. $\int_1^\infty \frac{\cos^2(x^3)+4}{x^{2/3}}\,dx$
4. $\int_0^1 \frac{e^x}{x}\,dx$

Differential Equations

Use Euler's method to compute the following estimates:

1. Initial value problem $y' = y + 2x$ with $y(1) = 0$, take 4 approximation steps to estimate $y(2)$.
2. Step size $h = 0.1$ to estimate $y(0.3)$ of $y' = y + xy$ with $y(0) = 1$.

Direction fields:
We have sketched the slope field for the differential equation $\frac{dy}{dt} = F(t, y)$ in the graphic below. Which of the four curves (labeled A, B, C, D) that we have drawn over the slope field could be the solution to this differential equation with the initial condition $y(0) = 1$?



Solve the initial value problem:

1. $\frac{dy}{dx} = xy^2, y(0) = 3$.

2. $xy^2y' = x+1, y(1) = 1$.
3. $\frac{dy}{dx} = xe^{-y}, y(2) = 1$.

Series

Tests for convergence:

(a) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$,

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$,

(c) $\sum_{k=1}^{\infty} \frac{1}{k^2+k^3}$,

(d) $\sum_{i=1}^{\infty} \frac{i}{\sqrt{i^5+1}}$,

(e) $\sum_{s=1}^{\infty} \frac{s^2-5s}{s^3+s+1}$,

(f) $\sum_{i=1}^{\infty} \frac{\ln(i)}{i}$,

(g) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n!}$,

(h) $\sum_{n=1}^{\infty} (-1)^{n+3} \frac{n^2}{n^3+4}$,

(i) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3}$,

(j) $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

(k) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

(l) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$

(m) $\sum_{n=1}^{\infty} \ln(\frac{1}{n}+1)$

Series arithmetic

1. Find the partial sum s_3 of the series $\sum_{n=1}^{\infty} 1/n^4$. Estimate the error in using s_3 as an approximation to the sum of the series.
2. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ up to two decimal places. Remark: the calculation of the final sum may be longer than what will be asked in an exam.
3. Harder: Determine the sums of $\sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{(2n)!}$, $\sum_{n=0}^{\infty} \frac{3^n}{2^n n!}$ and $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3}{2^{2n}}$. Hint: some may look like MacLaurin series of functions that you should remember. For the last series: try to split it into a positive and negative part to get two series that you recognise.
4. Use the MacLaurin Series of $\ln(x^2+1)$ to get a series with sum $\ln(1.25)$. How many terms of this series do we need to get an approximation with accuracy of 10^{-3}?

Find radius and Interval of Convergence:

1. $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$
2. $\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$
3. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$
4. $\sum_{n=1}^{\infty} \frac{n \cdot (x+1)^n}{4^n}$

Taylor and MacLaurin series

Give (quoting from textbook allowed!) the MacLaurin series and their radii of convergence for:

1. $\frac{1}{1-x}$
2. e^x
3. $\sin x$
4. $\cos x$

5. $\arctan x$
6. $\ln(1+x)$
7. $(1+x)^k$

Compute the MacLaurin series (by using the definition OR by subsituting into a known series) for the following functions:

1. e^{-2x^2}
2. $x \cdot \cos(x)$
3. $\cos(x^2)$
4. $x \cdot \cos(\frac{1}{2}x^2)$
5. $(1+2x)^{1/4}$

Use the MacLaurin series expansions to compute the following integrals:

1. $\int \sin(x^2)dx$
2. $\int \ln(1+x^2)dx$
3. $\int e^{3\sqrt{x}}dx$

What is the general formula of the MacLaurin series? What is the general formula for the Taylor Series at a point a?

Functions of several variables

Level Curves, Traces

Sketch the graph of the following functions (it is recommended to begin with determining the domain, the range, then sketching some well chosen level curves and traces).

1. $f(x,y) = xy + 2$.
2. $f(x,y) = \frac{1}{\sqrt{x^2+4y^2}}$.
3. $f(x,y) = \ln(y - 3x^2)$

Partial Derivatives

Find all second partial derivatives of f at the given points.

1. $f(x,y) = xy\sin(y^2)$ at $(3,2)$.
2. $f(x,y) = y^5 - 3x^2y$ at $(4,1)$.
3. $f(x,y) = \frac{x}{(x^2+y)^2}$ at $(1,1)$.
4. $f(x,y) = \sqrt{x}\ln(y^x)$ at $(1,4)$.

1. Find $f_{zyx} = \frac{\partial f}{\partial x \partial y \partial z}$ of $f(x,y,z) = 3xyz + x^2y^3z^7$. Evaluate at $(1,1,3)$.
2. Find $f_{yyx} = \frac{\partial f}{\partial x \partial y \partial y}$ of $f(x,y,z) = x^{\frac{12}{y}} + \ln(xy) + \sqrt{yz}$. Evaluate at $(1,2,3)$.

3. Let $u = e^{r\theta}\sin\theta$, find $\dfrac{\partial^3 u}{\partial r^2 \partial \theta}$.

Linear approximation:

1. Determine the linear approximation of the function $f(x, y) = xy\sin(y^2)$ at the point $(3, 2)$. Use this linear approximation to give an estimate for the value of $f(3.1, 2.02)$.
2. Determine the linear approximation of the function $f(x, y, z) = 3xyz + x^2y^3z^7$ at the point $(1, 1, 3)$. Use this linear approximation to give an estimate for the value of $f(0.9, 0.8, 3.01)$.

Chain rule

1. $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for $z = x^4 + x^2y$, $x = s + 2t$, $y = st$
2. $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ for $f(x, y) = \frac{x+y}{x-y}$, $x = 3st^2$, $y = s^2 + t^2$
3. For a function $z = (g(x), h(x))$ find $z'(7)$ with the given values:
$$g(7) = 3, \;\; h(7) = 2, \;\; g'(7) = -4, \;\; h'(7) = 1, \;\; \frac{\partial z}{\partial g}(3, 2) = -2, \;\; \frac{\partial z}{\partial h}(3, 2) = 9.$$
4. For a function $z = (t(u, v), s(u, v))$ find $\frac{\partial z}{\partial u}(2, 3)$ with the given values:
$$t(2, 3) = 4, \;\; s(2, 3) = 6, \;\; \frac{\partial t}{\partial u}(2, 3) = -4, \;\; \frac{\partial s}{\partial u}(2, 3) = 1, \;\; \frac{\partial z}{\partial t}(4, 6) = -2, \;\; \frac{\partial z}{\partial s}(4, 6) = 8.$$

Implicit differentiation and gradient vectors

Consider the function $z = f(x, y)$ defined implicitly by the equation $x^2z + xy - yz^3 = -1$.

1. Find the gradient vector of the function $z = f(x, y)$ at the point $(2, 1, -1)$.
2. Find the equation of the tangent plane of the graph of the equation at the point $(2, 1, -1)$.
3. Find the directional derivative of this function at the point $(2, 1, -1)$, in the direction of the vector $\mathbf{v} = (2, -3)$.
4. Find the maximum value of the directional derivative at $(2, 1, -1)$ among all possible directions.

Directional Derivatives

Beware: the directional vector may need to be of a certain length.

1. $f(x, y) = e^x\sin(y)$, $u = \langle 1, 5\rangle$ at $(0, \frac{\pi}{4})$
2. $f(x, y) = \frac{x}{x^2+y^2}$, approach from angle $\theta = \pi/3$, evaluate at $(2, 4)$
3. $f(x, y) = x^2y - xy$ from $u = 3\cdot\mathbf{i} + 6\cdot\mathbf{j}$. Recall that $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$.

Good luck with studying!