1341AF04 mid term solutions

1. It is known that a subspace Y of $\mathbf{R}^9$ can be spanned by 8 vectors, and that Y has a linearly independent set with 6 vectors. Then,

A. $\dim Y < 6$

B. $\dim Y > 6$

C. $6 < \dim Y \leq 8$

D. $6 \leq \dim Y < 8$

E. $6 \leq \dim Y \leq 8$

F. None of the above is true.

$*$ Size of a spanning set $\geq \dim Y \geq$ size of a l.i. set

($Y \subseteq \mathbb{R}^9$ so $\dim Y \leq \dim \mathbb{R}^9 =$

Hence $*$ implies

$6 \leq \dim Y \leq 8$

2. Suppose $\{u, v\}$ is a linearly **independent** set in vector space V, and that $w \in V$ is chosen so that $\{u, v, w\}$ is linearly **dependent**. Which of the following statements is always true?

A. $\{u, w\}$ is linearly dependent.

B. $\{v, w\}$ is linearly dependent.

C. $\{v, u\}$ is linearly dependent.

D. $u \in \operatorname{span}\{v, w\}$.

E. $v \in \operatorname{span}\{u, w\}$.

F. $w \in \operatorname{span}\{u, v\}$.

If $\{u, v\}$ is independent then $\{u, v, w\}$ is dependent iff $w \in \operatorname{span}\{u, v\}$.

3. If the augmented matrix of a linear system is $\begin{array}{c} \begin{matrix} & s & & t & & \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & 3 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right] \end{array}$, which of the following statements is true?

A. The system is inconsistent

B. $(-3s-6t, s, 2, t)$ is a solution for any values of s and t

C. $(-9, 1, 2, 1)$ is the unique solution of the system

D. $(3s+6t, s, -2, t)$ is a solution for any values of s and t

E. $(-3s-6t, s, 0, t)$ is a solution for any values of s and t

F. $(3s+6t, s, 0, t)$ is a solution for any values of s and t

$x_1 = -3s - 6t$

$x_2 = s$

$x_3 = 2$

$x_4 = t$

4. Let $U = \{(x, y, z) \in \mathbf{R}^3 \mid x - y + z = 0\}$.

$2\frac{1}{2}$ a) Find a basis for U and hence determine $\dim U$.

$1\frac{1}{2}$ b) Give a complete geometric description of U.

2 c) Extend your basis in (a) to a basis of $\mathbf{R}^3$.

a) $U = \{(x,y,z) \mid x-y+z=0\} = \{(x,y,z) \mid x = y-z\}$

$= \{(y-z, y, z) \mid y, z \in \mathbb{R}\} = \text{Span}\{(1,1,0), (-1,0,1)\}$.

Hence $\{(1,1,0), (-1,0,1)\}$ spans U. Moreover, $\{(1,1,0), (-1,0,1)\}$ is l.i. since the 2 vectors aren't parallel. Hence

$\{(1,1,0), (-1,0,1)\}$ is a basis of U. (1) - correct basis

(1) - just^n

Thus, $\dim U = 2$ $\left(\frac{1}{2}\right)$

(consistent with basis)

b) U is the plane through 0 with normal $(1,-1,1)$.

$1\frac{1}{2}$ (consistent with (a))

c) We need $v \in \mathbb{R}^3$ s.t. $v \notin \text{Span}\{(1,1,0), (-1,0,1)\}$

$= U$

So we choose any vector $v \notin U$, e.g.

$v = (1,0,0) \notin U$ since $1-0+0 \neq 0$

Then $\{(1,0,0), (1,1,0), (-1,0,1)\}$ is one such basis of $\mathbb{R}^3$ since it is l.i. and $\dim \mathbb{R}^3 = 3$

(1) - correct extension

5. Consider the vector space $\mathbf{F}[0,2] = \{f \mid f : [0,2] \to \mathbf{R}\}$. Suppose $f(x) = x$, $\quad g(x) = \frac{1}{x+1}$ and let

$$W = \operatorname{span}\{f, g\}.$$

2 a) Show that $\{f, g\}$ is linearly independent.

1 b) Find $\dim W$.

2 c) If $h(x) = x^2$, show that $h \notin W$.

1 d) What is the dimension of $\operatorname{span}\{f, g, h\}$?

a) Suppose $af + bg = 0$. Then $ax + \frac{b}{x+1} = 0$ * for all $x \in [0,2]$

(1)

at $x=0$, $* \Rightarrow 0 + \frac{b}{1} = 0$

at $x=1$, $* \Rightarrow a + \frac{b}{2} = 0$

$\Rightarrow \therefore b = 0, a = 0$ (1) (correct)

(at $x=2$, $* \Rightarrow 2a + \frac{b}{3} = 0$: not needed!) Hence, $\{f, g\}$ is l.i.

b) Since $\{f, g\}$ spans W (by defn of W!), and we know by (a) that $\{f, g\}$ is l.i., $\{f, g\}$ is a basis of W, so $\dim W = 2$.

($\frac{1}{2}$) – Correct dimension

($\frac{1}{2}$) – justification

c) Suppose $\underline{x^2 = ax + \frac{b}{x+1}}$ (**) for all $x \in [0,2]$ ($\frac{1}{2}$)

at $x=0$, we obtain $0 = 0 + b$

$x=1$, " $1 = a + \frac{b}{2}$

$x=2$, " $4 = 2a + \frac{b}{3}$

$\rightarrow a=1, b=0$

$\Downarrow$

$2a + \frac{b}{3} = 2 \neq 4$

Hence (**) is impossible (the system above is inconsistent)

$\therefore x^2 \notin \operatorname{span}\{x, \frac{1}{x+1}\}$.

($\frac{1}{2}$) correct dim ($\frac{1}{2}$) justn.

d) By (a) & (c), we know $\{f, g, h\}$ is l.i. (since $h \notin \operatorname{span}\{f,g\}$ &

$\frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

6. State whether the following are true or false. You must justify your answer: if true, explain why, if not, give an example to show it is false.

a) If V is a vector space and $\{v_1, v_2, v_3\} \subset V$ is linearly dependent, then $\{v_1, v_2\}$ is also linearly dependent.

False: e.g. $V = \mathbb{R}^2$, $v_1 = (1,0)$
$v_2 = (0,1)$
$v_3 = (0,0)$

Then $\{v_1, v_2, v_3\}$ is dependent but $\{v_1, v_2\}$ is l.i.

b) If u_1, u_2 and u_3 are non-zero vectors in a vector space V, and $U = \text{span}\{u_1, u_2, u_3\}$ then $\dim U = 3$.

False ($\{u_1, u_2, u_3\}$ may be dependent); e.g.

$V = \mathbb{R}^2$, $u_1 = (1,0)$, $u_2 = (0,1)$, $u_3 = (1,1)$!

Then $V = \text{span}\{u_1, u_2, u_3\}$ ($\{u_1, u_2, u_3\}$ is dependent

$= \mathbb{R}^2$ has dimension 2.

c) If W and X are subspaces of $\mathbf{R}^2$, then their intersection $W \cap X$ is also a subspace of $\mathbf{R}^2$.

True : subspace test ① $0 \in W$ and $0 \in X$ (they are s.s.)

$\therefore 0 \in W \cap X$

② If $u, v \in W \cap X$, then $u+v \in W$ (since $u, v \in W$ & W is a s.s)

and $u+v \in X$ (since $u, v \in X$ & X is a s.s.)

$\therefore u+v \in W \cap X$

③ If $k \in \mathbb{R}$, $u \in W \cap X$, then $ku \in W$ (since $u \in W$, W is a s.s)

and $ku \in X$ (since $u \in X$, X is a s.s)

$\therefore ku \in W \cap X$

$\therefore W \cap X$ is a s.s.

(OR : Consider 9 possibilities : $W = \mathbb{R}^2$, a line through 0, {0} & same for X.

d) Every linear system of 2 equations in 2 variables has a unique solution.

False i.e

$$x + y = 0$$
$$2x + 2y = 0$$

has infinitely many solns of the form $(-s, s)$, $s \in \mathbb{R}$

OR

$$x + y = 1$$
$$2x + 2y = 1$$

has no solutions at all.