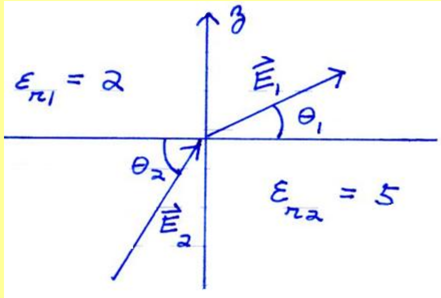
Boundary Condition Example



Given $\mathbf{E}_1 = 2\mathbf{x} - 3\mathbf{y} + 5\mathbf{z}$ V/m at a charge-free dielectric interface, find $\mathbf{D_2}$ and the angles θ_1 and θ_2.

Solution: The normal is +**z**, so the *z* components are the normal components. Since the interface is the *x*-*y* plane, both *x* and *y* components are tangential components. Thus:

$$\mathbf{E_1} = \quad =2\hat{\mathbf{x}}-3\hat{\mathbf{y}}+5\hat{\mathbf{z}}$$

$$\mathbf{E_2} = \quad =E_{2x}\hat{\mathbf{x}} + E_{2y}\hat{\mathbf{y}} + E_{2z}\hat{\mathbf{z}}$$

$$\mathbf{D_1} = \varepsilon_0\varepsilon_{r1}\mathbf{E_1} = 4\varepsilon_0\hat{\mathbf{x}} - 6\varepsilon_0\hat{\mathbf{y}} + 10\varepsilon_0\hat{\mathbf{z}}$$

$$\mathbf{D_2} = \quad =D_{2x}\hat{\mathbf{x}} + D_{2y}\hat{\mathbf{y}} + D_{2z}\hat{\mathbf{z}}$$

Applying $E_{1t} = E_{2t}$ gives:

$$E_{2x} = 2\ , \quad E_{2y} = -3$$

Applying $D_{1n} = D_{2n}$ gives:

$$D_{2z} = 10\varepsilon_0$$

The remaining components are found from $\mathbf{D_2} = \varepsilon\mathbf{E_2} = \varepsilon_0\varepsilon_{r2}\mathbf{E_2}$

$$D_{2x}\hat{\mathbf{x}} + D_{2y}\hat{\mathbf{y}} + 10\varepsilon_0\hat{\mathbf{z}} = 5\varepsilon_0\left(2\hat{\mathbf{x}} - 3\hat{\mathbf{y}} + E_{2z}\hat{\mathbf{z}}\right)$$

$$= 10\varepsilon_0\hat{\mathbf{x}} - 15\varepsilon_0\hat{\mathbf{y}} + 5\varepsilon_0 E_{2z}\hat{\mathbf{z}}$$

$$\Rightarrow D_{2x} = 10\varepsilon_0, \quad D_{2y} = -15\varepsilon_0, \quad E_{2z} = 2$$

Hence: $\mathbf{D_2} = 10\varepsilon_0\hat{\mathbf{x}} - 15\varepsilon_0\hat{\mathbf{y}} + 10\varepsilon_0\hat{\mathbf{z}}$

We now consider the angles made with the interface (the *x*-*y* plane):

$$\mathbf{E_1}\bullet\hat{\mathbf{z}} = |\mathbf{E_1}|\cos\left(90^\circ - \theta_1\right)$$

$$5 = \sqrt{(2)^2 + (-3)^2 + (5)^2}\cos(90^\circ - \theta_1)$$

$$5 = \sqrt{38}\sin\theta_1 \qquad \Rightarrow \qquad \theta_1 = \arcsin\left(0.811\right) = 54.2^\circ$$

Similarly: $\mathbf{E_2} \bullet \hat{\mathbf{z}} = |E_2| \cos(90^\circ - \theta_2)$

$2 = \sqrt{(2)^2 + (-3)^2 + (2)^2} \sin\theta_2 \quad \Rightarrow \quad \theta_2 = \arcsin(0.485) = 29.0^\circ$

Alternatively: Since $\tan\theta = \dfrac{|\mathbf{E_n}|}{|\mathbf{E_t}|}$, we have

$$\tan\theta_1 = \frac{E_{1z}}{\sqrt{E_{1x}^2 + E_{1y}^2}} = \frac{D_{1z}/\varepsilon_o\varepsilon_{r1}}{\sqrt{E_{1x}^2 + E_{1y}^2}}$$

$$\tan\theta_2 = \frac{E_{2z}}{\sqrt{E_{2x}^2 + E_{2y}^2}} = \frac{D_{2z}/\varepsilon_o\varepsilon_{r2}}{\sqrt{E_{2x}^2 + E_{2y}^2}}$$

Since $E_{1x} = E_{2x}$ and $E_{1y} = E_{2y}$, and $D_{1z} = D_{2z}$ …

Therefore: $\dfrac{\tan\theta_1}{\tan\theta_2} = \dfrac{\varepsilon_{r2}}{\varepsilon_{r1}}$

NB. This equation is related to the <u>field</u> directions, NOT the propagation directions....

i.e. This is <u>NOT</u> Snell's law!!