Module 2 Maxwell's Equations

Material in text: *Review*: Ch. 4 and Ch. 5

New: Ch. 6 (Sections 7 to 11) — *electrostatics* (Ch. 4) — *magnetostatics* (Ch. 5)

Modern electromagnetism is based on a set of four fundamental expressions that describe the time-dependent fields.

E – electric field [V/m]
D – electric displacement (electric flux density) [C/m^2]
H – magnetic field [A/m]
B – magnetic induction (magnetic flux density) [T]

In the time-dependent case, electric and magnetic fields are coupled – we cannot have one without the other.

We briefly review statics and then generalize to the dynamic case, highlighting differences.

Statics: all charges are fixed in space,
...or are moving at a constant velocity.

• *Electrostatics*

	Differential Form	Integral form
(1)	$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \bullet d\mathbf{l} = 0$
(2)	$\nabla \bullet \mathbf{D} = \rho_V$	$\oint_S \mathbf{E} \bullet d\mathbf{S} = \frac{Q}{\varepsilon}$

...where ρ_V is the volume charge density (C/m^3),
Q is the total charge enclosed in the volume V of surface S;
we take the medium (homogeneous) to be linear and isotropic.

We have the following constitutive relation...

(3) $\mathbf{D} = \varepsilon \mathbf{E}$ $\varepsilon = \varepsilon_o \varepsilon_r$ is the permittivity

...that brings in the material properties (likewise, the geometry, in the general case) of the medium.

We can define a scalar potential V such that

(4) $\mathbf{E} = -\nabla V$

Re-writing (2) as:

(5) $\nabla \bullet (\varepsilon \mathbf{E}) = \nabla \bullet (-\varepsilon \nabla V) = \rho_V$,

we get the scalar Poisson's equation:

(6) $\nabla^2 V = {}^{-\rho}\!/_{\varepsilon}$

In the absence of charge, this is LaPlace's equation:

(7) $\nabla^2 V = 0$

- *Magnetostatics*

Differential form | Integral form

(8) $\nabla \times \mathbf{B} = \mu \mathbf{J}$ $\qquad$ $\oint_C \mathbf{B} \bullet \mathbf{dl} = \mu I$

(9) $\nabla \bullet \mathbf{B} = 0$ $\qquad$ $\oint_S \mathbf{B} \bullet \mathbf{dS} = 0$

...where $\mathbf{J}$ is the current density (A/m^2), I is the total current through the surface S, and we presume a linear and isotroptic homogeneous medium.

We have the following constitutive relation

(10) $\mathbf{B} = \mu \mathbf{H}$ $\qquad$ $\mu = \mu_o \mu_r$ is the permeability

We can define a magnetic vector potential $\mathbf{A}$ ($\text{Webers}/\text{m} \equiv \text{Wb}/\text{m}$) :

(11) $\mathbf{B} = \nabla \times \mathbf{A}$

Substituting the above into (8) yields:

(12) $\nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J}$

Using the vector identity

(13) $\nabla \times \nabla \times \mathbf{A} = \nabla (\nabla \bullet \mathbf{A}) - \nabla^2 \mathbf{A}$

and the Coulomb gauge (more below)

(14) $\nabla \bullet \mathbf{A} = 0$

...generates the vector Poisson's equation:

(15) $\nabla^2 \mathbf{A} = -\mu \mathbf{J}$

Dynamic Fields

Charge and current sources vary with time

$\Rightarrow$ fields also vary with time

AND: electric and magnetic fields become interconnected

...coupling produces electromagnetic (EM) waves that propagate through media.

$\Rightarrow$ basis of the transport of energy and information

NB. In statics, an electric field in a conducting medium induces a permanent current which gives rise to a magnetic field. The static electric field is completely defined by the charge distribution and the magnetic field is a consequence. This is not the way it works in the dynamic case.

So, what is the relation between **E** and **H**
(or, equivalently, **D** and **B**)?

Equation (2) and (9) remain the same, but (1) and (8) need to be modified. Particularly, **E** is no longer conservative (the curl will now be non-zero), and it cannot be expressed as the function of a scalar potential (HOW one travels from point to point makes a difference).

Faraday's Law

If a current can produce a magnetic field then a magnetic field should produce a current…BUT the flux through the circuit must change with time...

(16) $$\nabla \times \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t}$$

• Integrating both sides over a surface *S* and applying Stoke's Theorem...

$$\int_S (\nabla \times \mathbf{F}) \bullet \mathbf{dS} = \oint_C \mathbf{F} \bullet \mathbf{dl}$$

yields:

(17) $$\oint_C \mathbf{E} \bullet \mathbf{dl} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \bullet \mathbf{dS}$$

For a stationary circuit, we can exchange integration and differentiation:

(18) $$\oint_C \mathbf{E} \bullet \mathbf{dl} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \bullet \mathbf{dS}$$

The LHS is merely the definition of the Emf (electromotive force) V (in volts):

(19) $$V = \oint_C \mathbf{E} \bullet \mathbf{dl}$$

The magnetic flux (in Webers) through S is:

$$\Phi = \int_S \mathbf{B} \bullet d\mathbf{S} \qquad (20)$$

Hence (18) becomes:

$$V = -\frac{\partial \Phi}{\partial t} \qquad (21)$$

Generalizing to a closed conducting loop of N turns:

$$V = -N \frac{\partial \Phi}{\partial t} \qquad (22)$$

Ampere's Law

$$\nabla \times \mathbf{H} = \mathbf{J} \qquad (23)$$ *Static case*

However, charge conservation yields

$$I = \oint_S \mathbf{J} \bullet d\mathbf{S} = \frac{-\partial Q}{\partial t} = \frac{-\partial}{\partial t} \int_V \rho_V \, dV \qquad (26)$$

Hence, by the Divergence theorem:

$$\nabla \bullet \mathbf{J} = \frac{-\partial \rho_V}{\partial t} \qquad (27)$$ *charge continuity equation*

Substituting (23) into the above generates an inconsistency:

(28) $\nabla \bullet (\nabla \times \mathbf{H}) = (\nabla \bullet \mathbf{J})$?!

is true only for the static case, since

$\nabla \bullet (\nabla \times \mathbf{F}) = 0$ is an identity!

If we add $+\partial\rho/\partial t$ to the RHS of (28), this will cancel the $-\partial\rho/\partial t$ $[cf.(27)]$ and the equation now works:

$$(29) \qquad \nabla \bullet (\nabla \times \mathbf{H}) = (\nabla \bullet \mathbf{J}) + \frac{\partial \rho_V}{\partial t} = 0$$

Recall that $\nabla \bullet \mathbf{D} = \rho_V$; if we then interchange space and time derivatives:

$$(30) \qquad \nabla \bullet (\nabla \times \mathbf{H}) = (\nabla \bullet \mathbf{J}) + \nabla \bullet \frac{\partial \mathbf{D}}{\partial t}$$

$$(31) \qquad \nabla \times \mathbf{H} = \mathbf{J} + \underbrace{\frac{\partial \mathbf{D}}{\partial t}}$$

displacement current density

Maxwell's equations – differential forms

$$(32) \qquad \nabla \times \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t}$$ Faraday's law

$$(33) \qquad \nabla \times \mathbf{H} = \mathbf{J} + \frac{-\partial \mathbf{D}}{\partial t}$$ Ampere's law

(34) $\nabla \bullet \mathbf{D} = \rho_V$ Gauss's law

(35) $\nabla \bullet \mathbf{B} = 0$ "no magnetic monopoles"

<u>Maxwell's equations – integral form</u>

(36) $$\oint_C \mathbf{E} \bullet \mathbf{dl} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \bullet \mathbf{dS} = \frac{-\partial \Phi}{\partial t}$$

(37) $$\oint_C \mathbf{H} \bullet \mathbf{dl} = \int_S \left\{ \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right\} \bullet \mathbf{dS} = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \bullet \mathbf{dS}$$

(38) $$\oint_S \mathbf{D} \bullet \mathbf{dS} = \int_V \rho_V dV = Q$$

(39) $$\oint_S \mathbf{B} \bullet \mathbf{dS} = 0$$

Let us take a deeper look. Combine (11) and (32):

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{and} \quad \nabla \times \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t}$$

Hence

(40) $$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A})$$

(41) $$\nabla \times \left\{ \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right\} = 0$$

Can our relation of field to scalar potential be modified? Set

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$$

That is, electric and magnetic fields can be written in terms of scalar and vector potentials:

(42) $$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

(42) Suggests that the electric field is composed of two terms:

* One part due to the charge distribution $\rho \rightarrow -\vec{\nabla} V$

This potential is deduced from either a...

(i) Volume charge distribution:

(43) $$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon} \int_V \frac{\rho_V(\mathbf{R}')}{R'} dV, \quad R' = |\mathbf{R} - \mathbf{R}'|$$

(ii) Surface charge distribution:

(44) $$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon} \int_S \frac{\rho_S(\mathbf{R}')}{R'} dS$$

(iii) Line charge distribution:

(45) $$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon} \int_C \frac{\rho_l(\mathbf{R}')}{R'} dl$$

*One part from the variation in current density (through $\partial \mathbf{D}/\partial t$):

$$\rightarrow -\frac{\partial \mathbf{A}}{\partial t}$$

From (15), the vector Poisson's equation $\nabla^2 \mathbf{A} = -\mu \mathbf{J}$, this gives

(46) $$\mathbf{A} = \frac{\mu}{4\pi}\int_V \frac{J(\mathbf{R}')}{R'}dV$$ (Wb/m) ...*cf*. (6) and (43)

One is tempted to naively add a time dependence

$$\text{eg. } V(\mathbf{R},t) = \frac{1}{4\pi\varepsilon}\int_V \frac{\rho_V(\mathbf{R}',t)}{R'}dV$$

This does not work, as it does not account for finite travel times.

eg. If V_1 is the potential due to ρ_1, then suddenly changing to ρ_2 would produce a sudden change to V_2, regardless of R

$\rightarrow$ This is unphysical.

We need to write the above as retarded potentials:

(47) $$V(\mathbf{R},t) = \frac{1}{4\pi\varepsilon}\int_V \frac{\rho_V\left(\mathbf{R}', t - R'/u_p\right)}{R'}dV$$

(48) $$\mathbf{A}(\mathbf{R},t) = \frac{\mu}{4\pi\varepsilon}\int_V \frac{\mathbf{J}\left(\mathbf{R}', t - R'/u_p\right)}{R'}dV$$

The reason for the above "kludge" is that the choice of potentials is not unique. We may have any **A** and *V* such that

(49a) $V \rightarrow V + \frac{\partial \Psi}{\partial t}$

(49b) $\vec{A} \rightarrow \vec{A} - \vec{\nabla}\Psi$

} Gauge transformation

...sub into (42)

where Ψ is completely arbitrary. This is because $\nabla \times (\nabla \Psi) = 0$ is an identity, and because space and time derivatives commute.

Consequences? Take...

$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}$ and Gauss's law $\vec{\nabla} \bullet \vec{E} = \rho / \varepsilon$

$$\Rightarrow -\nabla^2 V - \frac{\partial}{\partial t}\left(\vec{\nabla} \bullet \vec{A}\right) = \rho / \varepsilon$$

In electrostatics, we naturally chose $\nabla \bullet \mathbf{A} = 0$, yielding the scalar Poisson's equation

(50) $\nabla^2 V = -\rho / \varepsilon$

The separation of the field into charge induced and magnetically induced terms in NOT unique in the dynamic case. The appropriate choice is the "Lorentz gauge" ("velocity gauge")...

(51) $$\nabla \bullet \mathbf{A} = -\varepsilon\mu \frac{\partial V}{\partial t}$$

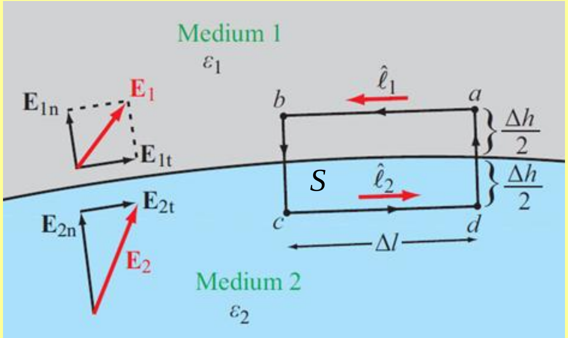
$$\Rightarrow -\nabla^2 V - \frac{\partial}{\partial t}\left\{-\varepsilon\mu \frac{\partial V}{\partial t}\right\} = \rho/\varepsilon$$

(52) $$\nabla^2 V - \frac{1}{u_p^2}\frac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon$$

This is the inhomogeneous wave equation, which will be also solved in the transmission line model.

Boundary Conditions

To solve EM problems, we need to know how the EM fields treat the border between dissimilar media. How are the field quantities in the first medium related to those in the second?



Consider the curl of the field across the patch *S*:

$$\int_S (\nabla \times \mathbf{E}) \bullet \mathbf{dS} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \bullet \mathbf{dS} = -\frac{\partial}{\partial t}\int_S \mathbf{B} \bullet \mathbf{dS} = -\frac{\partial \Phi}{\partial t} = \oint_C \mathbf{E} \bullet \mathbf{dl}$$

By Maxwell's equation, in the limit as $S \to 0$, the curl of **E** vanishes, because the flux through a vanishingly small area also vanishes; therefore by Stoke's theorem

$$\oint_C \mathbf{E} \bullet \mathbf{dl} = 0 \quad \Rightarrow \quad \int_a^b \mathbf{E}_2 \bullet \mathbf{dl} + \int_c^d \mathbf{E}_1 \bullet \mathbf{dl} = 0 \ \text{, as } \Delta h \to 0$$

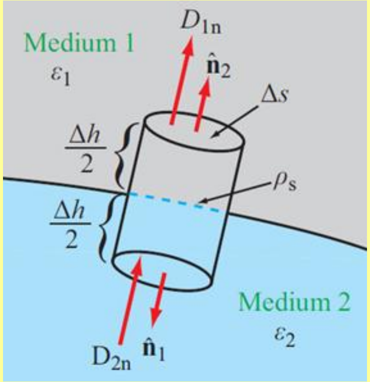
$$\therefore \quad E_{2t}\Delta l - E_{1t}\Delta l = 0 \quad \Rightarrow E_{1t} = E_{2t} \qquad (53)$$

The tangential component of the electric field is continuous across the boundary.

Since $D_{1t} = \varepsilon_1 E_1$ and $D_{2t} = \varepsilon_2 E_2$, the boundary condition on the tangential component of the electric flux density is:

$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2} \qquad (54)$$

What about the normal component?



Start with Gauss's law (integral form):

$$\oint_s \mathbf{D} \bullet \mathbf{dS} = Q$$

If $\Delta h \to 0$, then $Q = \int_V \rho_V dV \to \rho_S \Delta S$

$$\oint_{\substack{S \\ \Delta h \to 0}} \mathbf{D} \bullet \mathbf{dS} = \int_{top} \mathbf{D}_1 \bullet \hat{\mathbf{n}}_2 \mathbf{dS} + \int_{bottom} \mathbf{D}_2 \bullet \hat{\mathbf{n}}_1 \mathbf{dS}$$

NB. $\hat{\mathbf{n}}$ is an outward normal.

Since $\hat{\mathbf{n}}_1 = -\hat{\mathbf{n}}_2$, this becomes:

$$\hat{\mathbf{n}}_2 \bullet (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \;\Rightarrow\; D_{1n} - D_{2n} = \rho_s \qquad (55)$$

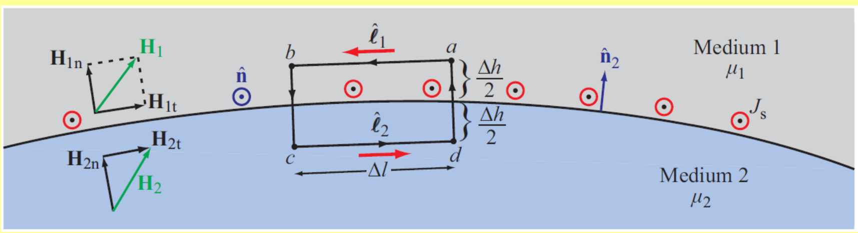
i.e., The normal component of **D** changes abruptly at a charged boundary.

We may treat the case of magnetic boundary conditions in a similar manner.

$$\text{Since } \oint_S \mathbf{D} \bullet d\mathbf{S} = Q \;\Rightarrow\; D_{1n} - D_{2n} = \rho_S$$

$$\text{then } \oint_S \mathbf{B} \bullet d\mathbf{S} = 0 \;\Rightarrow\; B_{1n} = B_{2n}$$

Hence: $$\mu_1 H_{1n} = \mu_2 H_{2n} \qquad (56)$$



For the tangential component, consider Ampere's law as applied to the above:

$$\oint_C \mathbf{H} \bullet d\mathbf{l} = I \;\Rightarrow\; \int_a^b \mathbf{H}_2 \bullet d\mathbf{l} + \int_c^d \mathbf{H}_1 \bullet d\mathbf{l} = I$$

$$H_{2t}\Delta l - H_{1t}\Delta l = J_S \Delta l \quad \text{or} \quad H_{2t} - H_{1t} = J_S \qquad (57)$$

Since surface currents only persist on the surfaces of perfect conductors, the J_s=0 for media with finite conductivity, and

$$H_{2t} = H_{1t} \qquad (58)$$

NB. For perfect dielectrics (zero conductivity, no free charges or Surface currents), we summarize:

$$D_{1n} = D_{2n} \quad , \quad B_{1n} = B_{2n}$$
$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} \; , \quad \mu_1 H_{1n} = \mu_2 H_{2n}$$

$$E_{1t} = E_{2t} \quad , \quad H_{1t} = H_{2t}$$
$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2} \; , \quad \frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}$$

More generally…

Field Components	General Form	Medium 1 Dielectric	Medium 2 Dielectric	Medium 1 Dielectric	Medium 2 Conductor
Tangential E	$\hat{\mathbf{n}}_2 \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$	$E_{1t} = E_{2t}$		$E_{1t} = E_{2t} = 0$	
Normal D	$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$	$D_{1n} - D_{2n} = \rho_s$		$D_{1n} = \rho_s$	$D_{2n} = 0$
Tangential H	$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$	$H_{1t} = H_{2t}$		$H_{1t} = J_s$	$H_{2t} = 0$
Normal B	$\hat{\mathbf{n}}_2 \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$	$B_{1n} = B_{2n}$		$B_{1n} = B_{2n} = 0$	

Notes: (1) ρ_s is the surface charge density at the boundary; (2) $\mathbf{J}_s$ is the surface current density at the boundary; (3) normal components of all fields are along $\hat{\mathbf{n}}_2$, the outward unit vector of medium 2; (4) $E_{1t} = E_{2t}$ implies that the tangential components are equal in magnitude and parallel in direction; (5) direction of $\mathbf{J}_s$ is orthogonal to $(\mathbf{H}_1 - \mathbf{H}_2)$.