

one point in time, swaps, discussed in Chapter 8, provide a price for a series of deliveries over time. Swaps are a natural generalization of forward contracts.

The pricing principles discussed in this chapter will also play important roles when we discuss option pricing in Chapters 10 and 11 and financial engineering in Chapter 12.

To get a sense of the range of traded contracts, look at the futures page of the *Wall Street Journal* and explore the Web sites of futures exchanges: the Chicago Mercantile Exchange Group ([www.cme.com](http://www.cme.com)), the New York Mercantile Exchange ([www.nymex.com](http://www.nymex.com)), and the London International Financial Futures Exchange ([www.liffe.com](http://www.liffe.com)), among others. These sites typically provide current prices, along with information about the contracts: what the underlying asset is, how the contracts are settled, and so forth. The site for OneChicago ([www.onechicago.com](http://www.onechicago.com)) provides information about single stock futures in the United States.

It is well accepted that forward prices are determined by the models and considerations in this chapter. Siegel and Siegel (1990) is a standard reference book on futures. Early papers that examined futures pricing include Modest and Sundaresan (1983), Cornell and French (1983), which emphasized tax effects in futures pricing, and French (1983), which compares forwards and futures when both exist on the same underlying asset. Brennan and Schwartz (1990) explore optimal arbitrage when there are transaction costs, and Reinganum (1986) explores the arbitrage possibilities inherent in time travel. There is a more technical academic literature focusing on the difference between forward and futures contracts, including Black (1976), Cox et al. (1981), Richard and Sundaresan (1981), and Jarrow and Oldfield (1981).

## PROBLEMS

- 5.1** Construct Table 5.1 from the perspective of a seller, providing a descriptive name for each of the transactions.
- 5.2** A \$50 stock pays a \$1 dividend every 3 months, with the first dividend coming 3 months from today. The continuously compounded risk-free rate is 6%.
- What is the price of a prepaid forward contract that expires 1 year from today, immediately after the fourth-quarter dividend?
  - What is the price of a forward contract that expires at the same time?
- 5.3** A \$50 stock pays an 8% continuous dividend. The continuously compounded risk-free rate is 6%.
- What is the price of a prepaid forward contract that expires 1 year from today?
  - What is the price of a forward contract that expires at the same time?
- 5.4** Suppose the stock price is \$35 and the continuously compounded interest rate is 5%.

- a. What is the 6-month forward price, assuming dividends are zero?
  - b. If the 6-month forward price is \$35.50, what is the annualized forward premium?
  - c. If the forward price is \$35.50, what is the annualized continuous dividend yield?
- 5.5** Suppose you are a market-maker in S&R index forward contracts. The S&R index spot price is 1100, the risk-free rate is 5%, and the dividend yield on the index is 0.
- a. What is the no-arbitrage forward price for delivery in 9 months?
  - b. Suppose a customer wishes to enter a short index futures position. If you take the opposite position, demonstrate how you would hedge your resulting long position using the index and borrowing or lending.
  - c. Suppose a customer wishes to enter a long index futures position. If you take the opposite position, demonstrate how you would hedge your resulting short position using the index and borrowing or lending.
- 5.6** Repeat the previous problem, assuming that the dividend yield is 1.5%.
- 5.7** The S&R index spot price is 1100, the risk-free rate is 5%, and the dividend yield on the index is 0.
- a. Suppose you observe a 6-month forward price of 1135. What arbitrage would you undertake?
  - b. Suppose you observe a 6-month forward price of 1115. What arbitrage would you undertake?
- 5.8** The S&R index spot price is 1100, the risk-free rate is 5%, and the continuous dividend yield on the index is 2%.
- a. Suppose you observe a 6-month forward price of 1120. What arbitrage would you undertake?
  - b. Suppose you observe a 6-month forward price of 1110. What arbitrage would you undertake?
- 5.9** Suppose that 10 years from now it becomes possible for money managers to engage in time travel. In particular, suppose that a money manager could travel to January 1981, when the 1-year Treasury bill rate was 12.5%.
- a. If time travel were costless, what riskless arbitrage strategy could a money manager undertake by traveling back and forth between January 1981 and January 1982?
  - b. If many money managers undertook this strategy, what would you expect to happen to interest rates in 1981?

- 10.2** Let  $S = \$100$ ,  $K = \$95$ ,  $r = 8\%$ ,  $T = 0.5$ , and  $\delta = 0$ . Let  $u = 1.3$ ,  $d = 0.8$ , and  $n = 1$ .
- Verify that the price of a European call is \$16.196.
  - Suppose you observe a call price of \$17. What is the arbitrage?
  - Suppose you observe a call price of \$15.50. What is the arbitrage?
- 10.3** Let  $S = \$100$ ,  $K = \$95$ ,  $r = 8\%$ ,  $T = 0.5$ , and  $\delta = 0$ . Let  $u = 1.3$ ,  $d = 0.8$ , and  $n = 1$ .
- Verify that the price of a European put is \$7.471.
  - Suppose you observe a put price of \$8. What is the arbitrage?
  - Suppose you observe a put price of \$6. What is the arbitrage?
- 10.4** Let  $S = \$100$ ,  $K = \$95$ ,  $\sigma = 30\%$ ,  $r = 8\%$ ,  $T = 1$ , and  $\delta = 0$ . Let  $u = 1.3$ ,  $d = 0.8$ , and  $n = 2$ . Construct the binomial tree for a call option. At each node provide the premium,  $\Delta$ , and  $B$ .
- 10.5** Repeat the option-price calculation in the previous question for stock prices of \$80, \$90, \$110, \$120, and \$130, keeping everything else fixed. What happens to the initial option  $\Delta$  as the stock price increases?
- 10.6** Let  $S = \$100$ ,  $K = \$95$ ,  $\sigma = 30\%$ ,  $r = 8\%$ ,  $T = 1$ , and  $\delta = 0$ . Let  $u = 1.3$ ,  $d = 0.8$ , and  $n = 2$ . Construct the binomial tree for a European put option. At each node provide the premium,  $\Delta$ , and  $B$ .
- 10.7** Repeat the option-price calculation in the previous question for stock prices of \$80, \$90, \$110, \$120, and \$130, keeping everything else fixed. What happens to the initial put  $\Delta$  as the stock price increases?
- 10.8** Let  $S = \$100$ ,  $K = \$95$ ,  $\sigma = 30\%$ ,  $r = 8\%$ ,  $T = 1$ , and  $\delta = 0$ . Let  $u = 1.3$ ,  $d = 0.8$ , and  $n = 2$ . Construct the binomial tree for an American put option. At each node provide the premium,  $\Delta$ , and  $B$ .
- 10.9** Suppose  $S_0 = \$100$ ,  $K = \$50$ ,  $r = 7.696\%$  (continuously compounded),  $\delta = 0$ , and  $T = 1$ .
- Suppose that for  $h = 1$ , we have  $u = 1.2$  and  $d = 1.05$ . What is the binomial option price for a call option that lives one period? Is there any problem with having  $d > 1$ ?
  - Suppose now that  $u = 1.4$  and  $d = 0.6$ . Before computing the option price, what is your guess about how it will change from your previous answer? Does it change? How do you account for the result? Interpret your answer using put-call parity.
  - Now let  $u = 1.4$  and  $d = 0.4$ . How do you think the call option price will change from (a)? How does it change? How do you account for this? Use put-call parity to explain your answer.

# Chapter 5

## Financial Forwards and Futures

### ■ Question 5.1

Four different ways to sell a share of stock that has a price  $S_0$  at time 0.

Description	Get Paid at Time	Lose Ownership of Security at Time	Receive Payment of
Outright Sale	0	0	$S_0$ at time 0
Security Sale and Loan Sale	$T$	0	$S_0 e^{rT}$ at time $T$
Short Prepaid Forward Contract	0	$T$	?
Short Forward Contract	$T$	$T$	$? \times e^{rT}$

### ■ Question 5.2

- The owner of the stock is entitled to receive dividends. As we will get the stock only in one year, the value of the prepaid forward contract is today's stock price, less the present value of the four dividend payments:

$$\begin{aligned}
 F_{0,T}^P &= \$50 - \sum_{i=1}^4 \$1e^{-0.06 \times \frac{3}{12}i} \\
 &= \$50 - \$0.985 - \$0.970 - \$0.956 - \$0.942 \\
 &= \$50 - \$3.853 = \$46.147
 \end{aligned}$$

- The forward price is equivalent to the future value of the prepaid forward. With an interest rate of 6% and an expiration of the forward in one year, we have:

$$\begin{aligned}
 F_{0,T} &= F_{0,T}^P \times e^{0.06 \times 1} = \$46.147 \times e^{0.06 \times 1} \\
 &= \$46.147 \times 1.0618 = \$49.00
 \end{aligned}$$

Note that this is equivalent to taking the future value of the initial stock price and subtracting the future value (at 6%) of the dividends received, as in Equation (5.6) of the text.

### ■ Question 5.3

1. The owner of the stock is entitled to receive dividends. We have to offset the effect of the continuous income stream in form of the dividend yield by tailing the position:

$$F_{0,T}^P = \$50e^{-0.08 \times 1} = \$50 \times 0.9231 = \$46.1558$$

We see that the value is very similar to the value of the prepaid forward contract with discrete dividends that we have calculated in Question 5.2. In Question 5.2., we received four cash dividends, with payments spread out through the entire year, totaling \$4. This implies a total annual dividend yield of approximately  $\$4 \div \$50 = 0.08$ .

2. The forward price is equivalent to the future value of the prepaid forward. With an interest rate of 6% and an expiration of the forward in one year we thus have:

$$\begin{aligned} F_{0,T} &= F_{0,T}^P \times e^{0.06 \times 1} = \$46.1558 \times e^{0.06 \times 1} \\ &= \$46.1558 \times 1.0618 = \$49.01 \end{aligned}$$

We could also use Equation (5.7) of the text, i.e.,

$$F_{0,T} = \$50e^{(0.06 - 0.08)} = \$49.01.$$

### ■ Question 5.4

1. We use the continuously compounded interest rate and the time to expiration in years (6 months is 0.5 year) in Equation (5.7). We have:

$$F_{0,T} = S_0 \times e^{r \times T} = \$35 \times e^{0.05 \times 0.5} = \$35 \times 1.0253 = \$35.886.$$

2. The annualized forward premium is calculated as:

$$\text{annualized forward premium} = \frac{1}{T} \ln \left( \frac{F_{0,T}}{S_0} \right) = \frac{1}{0.5} \ln \left( \frac{\$35.50}{\$35} \right) = 0.0284$$

Notice this is less than the interest rate, hence the index must pay a dividend.

3. We could use  $F_{0,T} = S_0 e^{(r-\delta)T}$  and solve for  $\delta$ . However, it is easier to use the previous result concerning the annualized forward premium. The forward premium is simply the difference between the risk-free rate and the dividend yield:

$$\begin{aligned} \text{annualized forward premium} &= \frac{1}{T} \ln \left( \frac{F_{0,T}}{S_0} \right) = \frac{1}{T} \ln \left( \frac{S_0 \times e^{(r-\delta)T}}{S_0} \right) \\ &= \frac{1}{T} \ln (e^{(r-\delta)T}) = \frac{1}{T} (r - \delta)T \\ &= r - \delta \end{aligned}$$

Therefore, we can solve:

$$\begin{aligned} 0.0284 &= 0.05 - \delta \Rightarrow \\ \delta &= 0.0216 \end{aligned}$$

The annualized dividend yield is 2.16%.

## ■ Question 5.5

1. We use the valuation formula,  $F_{0,T} = S_0 e^{(r-\delta)T}$ , with a continuously compounded interest rate of  $r = 5\%$ , a dividend yield of  $\delta = 0$ , and time to expiration  $T = 0.75$ . Remember time is in years, hence 9 months is  $3/4$  of a year. We have:

$$F_{0,T} = S_0 \times e^{r \times T} = \$1,100 \times e^{0.05 \times 0.75} = \$1,100 \times 1.0382 = \$1,142.02.$$

2. We engage in a reverse cash and carry strategy. In particular, we do the following:

Description	Today	In 9 months
Long forward, resulting from customer purchase	0	$S_T - F_{0,T}$
Sell short the index	$+S_0$	$-S_T$
Lend $+S_0$	$-S_0$	$S_0 \times e^{rT}$
Total	0	$S_0 \times e^{rT} - F_{0,T}$

Specifically, with the numbers given in the exercise, and assuming the forward price is the no arbitrage price we determined in Part (1), we have:

Description	Today	In 9 months
Long forward, resulting from customer purchase	0	$S_T - \$1,142.02$
Sell short the index	\$1,100	$-S_T$
Lend \$1,100	$-\$1,100$	$\$1,100 \times e^{0.05 \times 0.75}$ $= \$1,142.02$
Total	0	0

Therefore, the market maker is perfectly hedged. She does not have any risk in the future, because she has successfully created a synthetic short position in the forward contract.

3. Now, we will engage in cash and carry arbitrage:

Description	Today	In 9 months
Short forward, resulting from customer purchase	0	$F_{0,T} - S_T$
Buy the index	$-S_0$	$S_T$
Borrow $+S_0$	$+S_0$	$-S_0 \times e^{rT}$
Total	0	$F_{0,T} - S_0 \times e^{rT}$

Specifically, with the numbers of the exercise, we have:

Description	Today	In 9 months
Short forward, resulting from customer purchase	0	$\$1,142.02 - S_T$
Buy the index	$-\$1,100$	$S_T$
Borrow $\$1,100$	$\$1,100$	$-\$1,100 \times e^{0.05 \times 0.75}$ $= -\$1,142.02$
Total	0	0

Again, the market maker is perfectly hedged. He does not have any index price risk in the future, because he has successfully created a synthetic long position in the forward contract that perfectly offsets his obligation from the sold forward contract.

## ■ Question 5.6

- We use the valuation formula,  $F_{0,T} = S_0 e^{(r-\delta)T}$ , with a continuously compounded interest rate of  $r = 5\%$ , a dividend yield of  $\delta = .015$ , and time to expiration  $T = 0.75$ . We have:

$$F_{0,T} = S_0 \times e^{(r-\delta) \times T} = \$1,100 \times e^{(0.05-0.015) \times 0.75} = \$1,100 \times 1.0266 = \$1,129.26.$$

- We engage in a reverse cash and carry strategy. We must tail our short index position to  $e^{-\delta T} = e^{-0.015 \times 0.75} = 0.9888$ . Notice shorting the index requires paying the continuous dividends. Hence shorting 0.9888 units requires having to buy back more (i.e., 1 unit) of the index. The specifics of the reverse cash and carry are:

Description	Today	In 9 months
Long forward, resulting from customer purchase	0	$S_T - F_{0,T}$
Sell short tailed position of the index	$+S_0 e^{-\delta T}$	$-S_T$
Lend $S_0 e^{-\delta T}$	$-S_0 e^{-\delta T}$	$S_0 \times e^{(r-\delta)T}$
Total	0	$S_0 \times e^{(r-\delta)T} - F_{0,T}$

Using the given numbers, we have:

Description	Today	In 9 months
Long forward, resulting from customer purchase	0	$S_T - \$1,129.26$
Sell short tailed position of the index	$\$1,100 \times .9888$ $= 1087.69$	$-S_T$
Lend $\$1,087.69$	$-\$1,087.69$	$\$1,087.69 \times e^{0.05 \times 0.75}$ $= \$1,129.26$
Total	0	0

Therefore, the market maker is perfectly hedged. He does not have any risk in the future, because he has successfully created a synthetic short position in the forward contract.

3. With dividends reinvested, buying  $e^{\delta T}$  units today will grow to 1 unit in  $T$  years. The cash and carry table is:

Description	Today	In 9 months
Short forward, resulting from customer purchase	0	$F_{0,T} - S_T$
Buy tailed position in index	$-S_0 e^{-\delta T}$	$S_T$
Borrow $S_0 e^{-\delta T}$	$S_0 e^{-\delta T}$	$-S_0 \times e^{(r-\delta)T}$
Total	0	$F_{0,T} - S_0 \times e^{(r-\delta)T}$

Specifically, we have:

Description	Today	In 9 months
Short forward, resulting from customer purchase	0	$\$1,129.26 - S_T$
Buy tailed position in index	$-\$1,100 \times .9888$ $= -\$1,087.69$	$S_T$
Borrow $\$1,087.69$	$\$1,087.69$	$-\$1,087.69 \times e^{0.05 \times 0.75}$ $= -\$1,129.26$
Total	0	0

Again, the market maker is perfectly hedged. He does not have any index price risk in the future, because he has successfully created a synthetic long position in the forward contract that perfectly offsets his obligation from the sold forward contract.

## ■ Question 5.7

We need to find the fair value of the forward price first. With the numbers given, we have:

$$F_{0,T} = S_0 \times e^{r \times T} = \$1,100 \times e^{0.05 \times 0.5} = \$1,100 \times 1.02532 = \$1,127.85$$

1. If we observe a forward price of 1135, we know that the forward is too expensive, relative to the fair value we've determined. Therefore, we will short the forward contract at 1135, and create a long position in a synthetic forward (which has an implied price of 1,127.85). The synthetic long position is created by cash and carry. We should make a sure profit of \$7.15. We verify this with the following table:

Description	Today	In 9 months
Short forward	0	$\$1,135.00 - S_T$
Buy position in index	$-\$1,100$	$S_T$
Borrow $\$1,100$	$-\$1,100$	$\$1,127.85$
Total	100	$\$7.15$

This position requires no initial investment, has no index price risk, and has a strictly positive payoff. We have exploited the mispricing with a pure arbitrage strategy.

2. If we observe a forward price of 1115, we know that the forward is too cheap, relative to the fair value we have determined. Therefore, we will buy (i.e., go long) the forward at 1115, create a synthetic short forward at 1127.85, and make a sure profit of \$12.85. The synthetic short forward is achieved by engaging in a reverse cash and carry:

Description	Today	In 9 months
Long forward	0	$S_T - \$1,115.00$
Short position in index	\$1,100	$-S_T$
Lend \$1,100	-\$1,100	\$1,127.85
Total	0	\$12.85

This position requires no initial investment, has no index price risk, and has a strictly positive payoff. We have exploited the mispricing with a pure arbitrage strategy.

## ■ Question 5.8

We need to find the fair value of the forward price first. With the numbers given, we have:

$$F_{0,T} = S_0 \times e^{(r-\delta) \times T} = \$1,100 \times e^{(0.05-0.02) \times 0.5} = \$1,100 \times 1.01511 = \$1,116.62$$

1. If we observe a forward price of 1120, we know that the forward is too expensive, relative to the fair value we have determined. Therefore, we will sell the forward at 1120, and create a synthetic forward for 1116.62, making a sure profit of \$3.38. Our cash and carry has a tailed position due to the positive dividend yield:

Description	Today	In 9 months
Short forward	0	$\$1,120.00 - S_T$
Buy tailed position in index	$-\$1,100 \times .99$ $= -\$1,089.055$	$S_T$
Borrow \$1,089.055	\$1,089.055	$-\$1,116.62$
Total	0	\$3.38

This position requires no initial investment, has no index price risk, and has a strictly positive payoff. We have exploited the mispricing with a pure arbitrage strategy.

2. If we observe a forward price of 1110, we know that the forward is too cheap, relative to the fair value we have determined. Therefore, we will buy the forward at 1110, and create a synthetic short forward for 1116.62, thus making a sure profit of \$6.62. As we buy the real forward, we engage in a reverse cash and carry arbitrage (remembering to tail the position in order to cover dividends):

Description	Today	In 9 months
Long forward	0	$S_T - \$1,110.00$
Sell short tailed position in index	$\$1,100 \times .99$ $= \$1,089.055$	$-S_T$
Lend \$1,089.055	-\$1,089.055	\$1,116.62
Total	0	\$6.62

This position requires no initial investment, has no index price risk, and has a strictly positive payoff. We have exploited the mispricing with a pure arbitrage strategy.

**■ Question 5.9**

1. A money manager could take a large amount of money, travel back to 1981, invest it at 12.5%, and instantaneously travel forward to 1982 to reap the benefits, i.e., the accrued interest. Then repeat the process as many times as possible. Our argument of time value of money breaks down.
2. If many money managers undertook this strategy, the large amount of funds they'd want to invest at 12.5% would drive the 1981 interest rates down.
3. Unfortunately, these arguments mean that costless and riskless time travel will not be invented. The same point can be made using almost any other financial asset.

## ■ Question 10.2

1. We will use risk-neutral pricing for this problem. Our risk-neutral probability of an up movement is

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{0.04} - .8}{1.3 - .8} = 0.4816$$

The no-arbitrage value of the European call is:

$$C_0 = e^{-rh}[C_u p^* + C_d(1 - p^*)] = e^{-0.04} \times 35 \times 0.4816 = 16.1958$$

For the replicating strategy, the option's delta is  $\Delta = 35/50 = 0.70$ . We can use the option value result to solve for the bond amount (once again, no memorization required). The option value is the total cost of replication. The cost of 0.7 shares is \$70. Since the option value is \$16.1958, we must borrow  $\$70 - \$16.1958 = \$53.8042$ , i.e.,  $B = -\$53.8042$ .

2. If we observe a price of \$17, then the option price is too high relative to its theoretical value. We sell the option for \$17 and synthetically create a call option for \$16.196. In order to do so, we buy 0.7 units of the share and borrow \$53.804. These transactions yield no risk and a profit of \$0.804.

3. If we observe a price of \$15.50, then the option price is too low relative to its theoretical value. We buy the option and synthetically create a short position in the option (we'll receive \$16.1958). In order to do so, we short 0.7 shares and lend \$53.8042. These transactions yield no risk and a profit of \$0.696.

### ■ Question 10.3

1. We will use risk-neutral pricing for this problem. Our risk-neutral probability of an up movement is

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{0.04} - .8}{1.3 - .8} = 0.4816$$

The no-arbitrage value of the European put is:

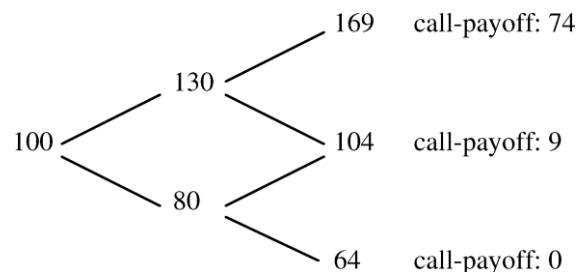
$$P_0 = e^{-rh}[P_u p^* + P_d(1 - p^*)] = e^{-.04} \times 15 \times 0.5184 = 7.4708$$

For the replicating strategy, the option's delta is  $\Delta = -15/50 = -0.30$ . We can use the option value result to solve for the bond amount (once again, no memorization required). The option value is the total cost of replication. By shorting 0.3 shares, we receive \$30. Since the option value is \$7.4708, we must lend \$37.4708, i.e.,  $B = \$37.4708$ .

2. If we observe a price of \$8, then the option price is too high relative to its theoretical value. We sell the option for \$8 and synthetically create a long put option for \$7.471. In order to do so, we short 0.3 units of the share and lend \$37.4708. These transactions yield no risk and a profit of \$0.5292.
3. If we observe a price of \$6, then the put option price is too low relative to its theoretical value. We buy the option and synthetically create a short position in the option (we'll receive \$7.4708). In order to do so, we buy 0.3 units of the share and borrow \$37.4708. These transactions yield no risk and a profit of \$1.4708.

### ■ Question 10.4

The stock prices evolve according to the following picture:



Since we have two binomial steps, and a time to expiration of one year,  $h$  is equal to 0.5. Therefore, we can calculate with the usual formulas for the respective nodes:

$t = 0, S = 100$	$t = 1, S = 80$	$t = 1, S = 130$
$\Delta = 0.691$	$\Delta = 0.225$	$\Delta = 1$
$B = -49.127$	$B = -13.835$	$B = -91.275$
$C_0 = 19.994$	$C_d = 4.165$	$C_u = 38.725$