Print Name :

Student Number:

Section (either A, B, C, or D. See above for your Instructor's name):

PART A

Do all (12) Questions for a total of 36 marks out of a maximum of 80.
Do not detach nor unstaple this examination. Missing sheets will void any credit for those questions.

- **A1.** The slope of the tangent line to the cardioid $r = 1 + \cos\theta$ at $\theta = \frac{\pi}{6}$ is

  (a) -1 (b) $\frac{\sqrt{3}}{2}$ (c) horizontal (d) $\frac{1}{2}$ (e) vertical

  Solution: (a)

- **A2.** The angle α between the two planes $x + 2y + z = 1$ and $2x - 3y + 4z = 3$ is

  (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) π (d) 0 (e) $\frac{\pi}{2}$

  Solution: (e)

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- **A3.** Compute the volume of the parallelepiped defined by the vectors $< 1, 2, 3 >$, $< 1, -1, 1 >$ and $< 3, 1, -2 >$.
  (**Note:** In this examination the symbols (a, b, c) and $< a, b, c >$ both denote the vector with components a, b, c, etc.)

  (a) 12 (b) 23 (c) -12 (d) 8 (e) 13

  Solution: (b)

- **A4.** Calculating the divergence of $\vec{E} =< e^{-x}\cos y,\ e^{x}\sin y,\ \ln z >$ at the point $(0, \frac{\pi}{3}, 5)$ yields

  (a) $\frac{\ln 5}{2}$ (b) $\frac{\sqrt{3}}{2}\ln 5$ (c) 5 (d) 5π (e) $\frac{1}{5}$

  Solution: (e)

- **A5.** Let $w(x, y) = yx^2 - xy^3$, where $x = \cos t, y = e^t$. Use the Chain Rule to evaluate $\frac{\partial w}{\partial t}$ at $t = 0$:

  (a) 1 (b) e (c) 4 (d) -2 (e) None

  Solution: (d)

- **A6.** A vector giving the direction in which $f(x, y) = \sqrt{2x + 3y^2}$ increases most rapidly at the point (0,-1) is given by

  (a) $< -\frac{3}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$ (b) $< \frac{\sqrt{3}}{3}, 3 >$ (c) $< 1, -3 >$ (d) $\frac{1}{\sqrt{3}} < 2, 3 >$ (e) None

  Solution: (c)

- **A7.** Given that $(0,0)$ and $(2,1)$ are critical points of the function $f(x,y) = x^3 - 12xy + 8y^3$. Then
  a: f has a local maximum at $(2,1)$ and local minimum at $(0,0)$
  b: f has a local minimum at $(2,1)$ and local maximum at $(0,0)$
  c: f has a local minimum at $(2,1)$ and $(0\ ,\ 0)$ is a saddle point
  d: f has a local maximum at $(2,1)$ and $(0\ ,\ 0)$ is a saddle point
  e: None

  Solution: (c)

- **A8.** Evaluate the double integral $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx$.

  (a) $\frac{4\pi}{3}$ (b) $\frac{8\pi}{3}$ (c) $\frac{16\pi}{3}$ (d) $\frac{32\pi}{3}$ (e) None
  Solution: (b)

- **A9.** Evaluate the triple integral $\displaystyle\iiint_{\mathcal{T}} 3z^2y\,dz\,dy\,dx$ where $\mathcal{T}$ is the region

  $\mathcal{T} = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le \sqrt[3]{x+y}\}$.

  (a) $\frac{7}{12}$ (b) $\frac{1}{3}$ (c) $\frac{3}{8}$ (d) $\frac{3}{14}$ (e) None
  Solution: (a)

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- **A10.** Let the triple integral $I = \displaystyle\iiint_E \sqrt{x^2+y^2+z^2}\,dV$, where E lies between the spheres $x^2+y^2+z^2 = 4$ and $x^2+y^2+z^2 = 9$, in the first octant, be evaluated by using the spherical coordinates. Then $I =$

$$a = \int_0^{\pi}\int_0^{\frac{\pi}{4}}\int_2^{3} \rho^3 \sin\phi\,d\rho\,d\theta\,d\phi \qquad b = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_2^{2} \rho^3 \sin\phi\,d\rho\,d\theta\,d\phi$$

$$c = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_2^{3} \rho^3 \sin\phi\,d\rho\,d\theta\,d\phi \qquad d = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_1^{2} \rho \sin\phi\,d\rho\,d\theta\,d\phi$$

$$e = \int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_4^{9} \rho^3 \sin\phi\,d\rho\,d\theta\,d\phi$$

  Solution: (c)

- **A11.** The line integral $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$ and $C:\ \mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + t^3\mathbf{k},\ \ 0 \le t \le 1$ is:

  (a) 1 (b) 2 (c) 3 (d) 4 (e) None
  Solution: (b)

- **A12.** Let $\mathbf{F}(x,y,z) = 3x\mathbf{i} + 3y\mathbf{j} + z\mathbf{k}$. Evaluate the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$ where S is the open surface bounded by the paraboloid $z = 6 - x^2 - y^2$ lying above the plane $z = 2$ (this means that this part of $z = 2$ is excluded from the surface). Here, $\mathbf{n}$ is an outer normal unit vector to S.

  (a) 24π (b) 36π (c) 0 (d) 64π (e) None of these

End of Part A

Do All six (6) Questions for a total of 44 marks out of a maximum of 80.

Do not detach nor unstaple this examination. Missing sheets will void any credit for those questions.

- **B1.** [5+ 3 marks] Find the extrema of the function $f(x,y) = x + 2y$ subject to the constraint $y^2 + xy - 1 = 0$ and classify them as a max/min [3 marks].

Solution: $(0, \pm 1)$. This is because $F(x,y,\lambda) = x + 2y - \lambda(y^2 + xy - 1)$ must satisfy

$$\begin{aligned} \frac{\partial F}{\partial x} &= 1 - \lambda y = 0 \\ \frac{\partial F}{\partial y} &= 2 - 2\lambda y - \lambda x = 0 \\ \frac{\partial F}{\partial \lambda} &= y^2 + xy - 1 = 0 \end{aligned}$$

The third equation implies that $y \neq 0$ so that from the first of these we get that $\lambda = 1/y$. Using this relation in the second equation we get that $x/y = 0$ or $x = 0$. Finally, substituting the latter into the third equation once again we get that $y^2 = 1$ or $y = \pm 1$. It follows that $(x,y) = (0, \pm 1)$ are the extrema. ($f(0,1) = 2$ being a maximum and $f(0,-1) = -2$ being the minimum.)

- **B2.** [7 marks] Evaluate the double integral $I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-4\,(x^2+y^2)}\,dx\,dy$, by a suitable change of coordinates.

Solution: Change to polar coordinates on the plane, $x = r\cos\theta, y = r\sin\theta$. Then $dx\,dy = r dr\,d\theta$ and the region of integration becomes $\mathcal{R} = \{(r,\theta) : 0 \le r < \infty, 0 \le \theta \le 2\pi\}$.

Then

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-4\,(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} re^{-4r^2}\,dr\,d\theta = 2\pi\int_0^{\infty} re^{-4r^2}\,dr = \frac{2\pi}{8} = \frac{\pi}{4}.$$

- **B3.** [7 marks] Evaluate the line integral $I = \int_C y\,ds$ over the arc C defined parametrically by $x = 2t$, $y = t^3$, $0 \le t \le 1$.

Solution: Here $ds = \sqrt{x'(t)^2 + y'(t)^2}\,dt = \sqrt{4 + 9t^4}\,dt$. Since $y = t^3$ it follows that

$$I = \int_C y\,ds = \int_0^1 t^3\sqrt{4+9t^4}\,dt = \frac{1}{54}(13\sqrt{13} - 8).$$

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- **B4.** [7 marks] Determine whether the given vector field $\mathbf{F}(x, y, z) = 2y^2z^3\,\mathbf{i} + 4xyz^3\,\mathbf{j} + 6xy^2z^2\,\mathbf{k}$ is conservative and if so, find a potential function $f(x, y, z)$ such that $\nabla f = \mathbf{F}$.

Solution: Yes, since $\operatorname{curl}\mathbf{F} = \mathbf{0}$. $f(x, y, z) = 2xy^2z^3 + C$ where C is an arbitrary constant.

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- **B5.** [7 marks] Let $\mathcal{C}$ denote the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ oriented positively. Evaluate $\displaystyle\int_{\mathcal{C}} (x + e^x \sin y)\,dx + (x + e^x \cos y)\,dy$ either directly or by using some theorem. (Hint: The area of an ellipse with axes a, b is given by πab.)

Solution: Use Green's Theorem. Since $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$ it follows from Green's Theorem that $\displaystyle\int_{\mathcal{C}} (x + e^x \sin y)\,dx + (x + e^x \cos y)\,dy = \iint_{\mathcal{R}} dA$ where $\mathcal{R}$ is the region defined by the ellipse in the hypothesis. Since $\iint_{\mathcal{R}} dA$ is the area of the ellipse it follows that $\displaystyle\int_{\mathcal{C}} (x + e^x \sin y)\,dx + (x + e^x \cos y)\,dy = \pi \cdot 2 \cdot 3 = 6\pi$.

- **B6.** [8 marks] Let $\mathbf{F}(x,y,z) = 2y\,\mathbf{i} + xz^2\,\mathbf{j} + x^2ye^z\,\mathbf{k}$ and let $\mathcal{S}$ denote the hemisphere $z = \sqrt{4 - x^2 - y^2}$. Evaluate the surface integral $\displaystyle\iint_{\mathcal{S}} \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS$ using any method. Here $\mathbf{n}$ is an outer normal unit vector to $\mathcal{S}$.

Solution: Use Stokes' Theorem. Thus,

$$\iint_{\mathcal{S}} \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r},$$

where $\mathcal{C}$ is the circle $x^2+y^2=4$ on the plane $z=0$ oriented positively. Using the parametrization $x = 2\cos t, y = 2\sin t$, $0 \le t \le 2\pi$ for $\mathcal{C}$ we find that

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi} (4\sin t, 0, 8\cos^2 t\sin t)\cdot(-2\sin t, 2\cos t, 0)\,dt \\
&= -8\int_0^{2\pi} \sin^2 t\,dt \\
&= -8\pi.
\end{aligned}$$