University of Ottawa
Department of Mathematics and Statistics
Calculus III for Engineers
MAT 2322 X00 - Spring-Summer 2019
Midterm II - V.1 – July 11
Professor: Abdelkrim El basraoui

Solutions

Name:______________________________ ID Number:____________________

Instructions: (Please read carefully.)

- This exam has 10 pages and 6 questions, and you have 80 minutes to complete it.
- This is a closed book exam.
- **The only calculators which are allowed are those approved by the faculty of science such as Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
- Questions 1 and 2 are worth 5 marks each, Questions 3 and 4 are worth 4 marks each, and Questions 5 and 6 are worth 6 marks, so organize your time accordingly.
- Answer each question in the space provided or using backs of pages if necessary.
- Page 8 is an extra page for Question 5.
- **A correct answer requires a full, clearly-written and detailed solution.**
- Do not unstaple the test.
- Good luck!
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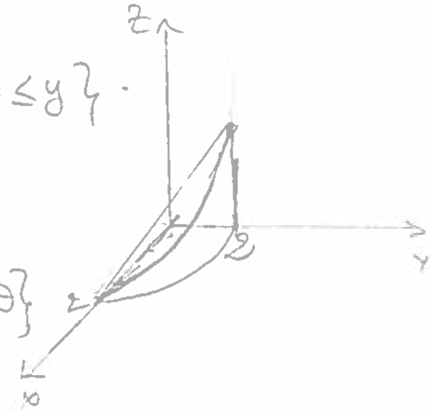
Signature: ______________________________________

Question	Q.1	Q.2	Q.3	Q.4	Q.5	Q.6	Total
Maximum	5	5	4	4	6	6	30
Score							

1. Consider the solid **in the first octant** that lies within the cylinder $x^2 + y^2 = 4$ below the plane $z = y$. This solid has a mass density given by the function $\delta(x, y, z) = x$. Find the total mass of this solid.



Here $E = \{(x,y,z) \mid x^2+y^2 \leq 4,\ x \geq 0, y \geq 0,\ 0 \leq z \leq y\}$.

In cylindrical coords

$$E = \{(r,\theta,z) \mid 0 \leq r \leq 2,\ 0 \leq \theta \leq \frac{\pi}{2},\ 0 \leq z \leq r\sin\theta\}$$

& $\delta(r,\theta,z) = r\cos\theta$.

So, the mass is

$$m = \iiint_E x\, dV = \int_0^2 \int_0^{\pi/2} \int_0^{r\sin\theta} r\cos\theta\, \underbrace{r\,dz\,d\theta\,dr}_{dV \text{ in cylindrical coords.}}$$

$$= \int_0^2 \int_0^{\pi/2} r^3 \cos\theta \sin\theta\, d\theta\, dr$$

$$= \int_0^2 \frac{r^3}{2} \sin^2\theta \Big|_0^{\pi/2} dr$$

$$= \int_0^2 \frac{r^3}{2}\, dr$$

$$= \frac{r^4}{8}\Big|_0^2$$

$$= \boxed{2}$$

P.S.: One may use $E = \{(x,y,z) \mid 0 \leq x \leq 2,\ 0 \leq y \leq \sqrt{4-x^2},\ 0 \leq z \leq y\}$, however you'd need to convert to polar coords to evaluate the double integral

2. Consider the vector field $\vec{F} = \langle xy, x \rangle$, and let C be the upper half-circle $x^2 + y^2 = 1$ oriented from (1,0) to (-1,0). Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r}.$$

• Parametrization : $\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}, \quad 0 \le \theta \le \pi$

so that $\vec{r}(\theta) = \langle \cos\theta, \sin\theta \rangle, \quad 0 \le \theta \le \pi,$

$\Rightarrow \vec{r}\,'(\theta) = \langle -\sin\theta, \cos\theta \rangle$

& $\vec{F}(\vec{r}(t)) = \langle \cos\theta \sin\theta, \cos\theta \rangle$

• Therefore

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} \langle \cos\theta \sin\theta, \cos\theta \rangle \cdot \langle -\sin\theta, \cos\theta \rangle \, d\theta$$

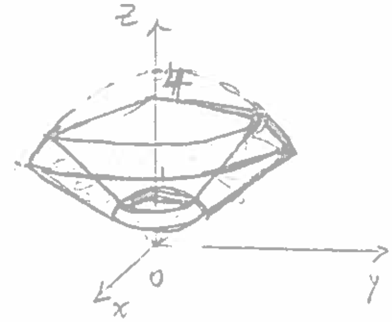
$$= \int_0^{\pi} \left(-\cos\theta \sin^2\theta + \cos^2\theta \right) d\theta$$

$$= \int_0^{\pi} \left(-\cos\theta \sin^2\theta + \frac{1+\cos(2\theta)}{2} \right) d\theta$$

$$= \left(-\frac{\sin^3\theta}{3} + \frac{\theta}{2} - \frac{\sin(2\theta)}{4} \right) \Big|_0^{\pi}$$

$$= \boxed{\frac{\pi}{2}}$$

3. Consider the solid **in the first octant** which lies in between the cones $z = \frac{\sqrt{3}}{3}\sqrt{x^2+y^2}$ and $z = \sqrt{x^2+y^2}$, and the spheres $x^2+y^2+z^2 = 1$, and $x^2+y^2+z^2 = 16$. This solid has a mass density given by the function $\delta(x,y,z) = z^2$. Set up a triple integral in **spherical coordinates** which gives the total mass of this solid. **DO NOT EVALUATE THE INTEGRAL.**



(FYI: the angles that make the cones $z = \frac{\sqrt{3}}{3}\sqrt{x^2+y^2}$ & $z = \sqrt{x^2+y^2}$ with the positive z-axis are respectively $\frac{\pi}{3}$ & $\frac{\pi}{4}$)

Here $E = \{(x,y,z) \mid 1 \le x^2+y^2+z^2 \le 16,\ \frac{\sqrt{3}}{3}\sqrt{x^2+y^2} \le z \le \sqrt{x^2+y^2},\ x \ge 0,\ y \ge 0,\ z \ge 0\}$

In spherical coords

$E = \{(\rho,\theta,\phi) \mid 1 \le \rho \le 4,\ 0 \le \theta \le \frac{\pi}{2},\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\}$,

& $\delta(\rho,\theta,\phi) = \rho^2 \cos^2\phi$

Therefore, the total mass is

$$\boxed{m = \int_1^4 \int_0^{\pi/2} \int_{\pi/4}^{\pi/3} \rho^2 \cos^2\phi \ \overbrace{\rho^2 \sin\phi \, d\phi \, d\theta \, d\rho}^{dV \text{ in spherical coords.}}}$$

4. Find the total arclength of the curve which is parametrized by the following vector function

$$\vec{r}(t) = \sin(t)\,\vec{\mathbf{i}} + \cos(t)\,\vec{\mathbf{j}} + 4t\,\vec{\mathbf{k}}, \quad 0 \le t \le \pi/4.$$

$\Longrightarrow \vec{r}\,'(t) = \langle \cos t, -\sin\theta, 4 \rangle$

& $\|\vec{r}\,'(t)\| = \sqrt{\cos^2\theta + \sin^2\theta + 16}$

$= \sqrt{17}$

$\therefore \; L = \int_0^{\pi/4} \|\vec{r}\,'(t)\|\,dt = \boxed{\dfrac{\pi\sqrt{17}}{4}}$

5. *Part (a) is independent of parts (b) and (c).*

(a) Determine which among the following vector fields are conservative vector fields. Justify your answer.

$$\vec{F} = y^2 \cos x\vec{\mathbf{i}} - 2y \sin x\vec{\mathbf{j}}$$
$$\vec{G} = 2xe^{-y}\vec{\mathbf{i}} + (2y - x^2e^{-y})\vec{\mathbf{j}} + 2z\vec{\mathbf{k}}$$
$$\vec{H} = (\cos(y) + x)\vec{\mathbf{i}} - x\sin(y)\vec{\mathbf{j}}$$
$$\vec{K} = \cos(xz)\vec{\mathbf{i}} + \sin(yz)\vec{\mathbf{j}} + xy\sin(z)\vec{\mathbf{k}}$$

(b) Find a potential function f for the conservative vector field $\vec{F} = y\sin(x)\vec{\mathbf{i}} - \cos(x)\vec{\mathbf{j}}$.

(c) Evaluate the line integral of **the conservative vector field** in (b) along the curve C, where C is any smooth curve from $(0,0)$ to $(2\pi, 1)$.

(a) • For $\vec{F}$, $P(x,y) = y^2\cos x$ & $Q(x,y) = -2y\sin x$.

So, $P_y = 2y\cos x \neq Q_x = -2y\cos x$ & thus $\vec{F}$ is not conservative

• For $\vec{H}$, $P(x,y) = (\cos y + x)$ & $Q(x,y) = -x\sin y$.

So, $P_y = -\sin y = Q_x$. & thus $\vec{H}$ is conservative.

• For $\vec{G}$, we have $\text{Curl } \vec{G} = \nabla \times \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xe^{-y} & (2y - x^2e^{-y}) & 2z \end{vmatrix}$

$= \langle 0, 0, 0 \rangle$

& thus $\vec{G}$ is conservative.

• For $\vec{K}$, we have $\text{Curl } \vec{K} = \nabla \times \vec{K} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \cos(xz) & \sin(yz) & xy\sin z \end{vmatrix}$

$= \langle x\sin z - y\cos(yz), -x\sin(xz) - y\sin z, 0 \rangle$
$\neq \hat{0}$

& thus $\vec{K}$ is not conservative

Extra page for Question 5. v.

(b) Let $f(x,y)$ be such that $\vec{F} = \nabla f \iff \begin{cases} f_x = y\sin x \\ f_y = -\cos x \end{cases}$

Then $f(x,y) = \int y \sin x \, dx$ $\left(\underline{\text{or}} \int -\cos x \, dy\right)$

$= -y\cos x + g(y)$ $\left(\overset{\text{or}}{=} -y\cos x + h(x)\right)$

But $f_y = -\cos x \iff -\cos x + \frac{d}{dx}g(x) = \cos x$ (or $f_x = y\sin x$

$\iff \frac{d}{dx}g(x) = 0 \implies g(x) = K,\ K \in \mathbb{R}$.

Take $K = 0$.

$\iff y\sin x + \frac{d}{dx}h(x) = y\sin x$

$\iff \frac{d}{dx}h(x) = 0$

$\implies h(x) = K,\ K \in \mathbb{R}$

Take $K = 0$)

$\therefore \boxed{f(x,y) = -y\cos x}$

(c) $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} \overset{\text{F.T.L.I}}{=} f(2\pi, 1) - f(0,0)$

$= -(1)\cos(2\pi) - 0$

$= \boxed{-1}$

6. Use Green's Theorem to evaluate the line integral $\oint_C y\,dx + x^2y\,dy$, where C , where C is the unit circle with counterclockwise orientation (which is positive orientation).

Here $\vec{F} = \langle P(x,y) = y,\ Q(x,y) = x^2 y \rangle$.

So, $Q_x - P_y = 2xy - 1$ ($= 2r^2\cos\theta\sin\theta - 1$ in polar coords)

Also, since C is the unit circle (with ccw orientation), D is the unit disk $D = \{(x,y) \mid x^2+y^2 \le 1\}$

($= \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$ in polar coords)

By Green's Thm, we have

$$\oint_C y\,dx + x^2y\,dy = \iint_D (Q_x - P_y)\,dA$$

$$= \iint_D (2xy-1)\,dA$$

In polar coords

$$= \int_0^1 \int_0^{2\pi} (2r^2 \sin\theta\cos\theta - 1)\, \overbrace{r\,d\theta\,dr}^{dA \text{ in polar coord}}$$

$$= \int_0^1 r\left(r^2\sin^2\theta - \theta\right)\Big|_0^{2\pi} dr$$

$$= \int_0^1 -2\pi r\,dr$$

$$= -\pi r^2 \Big|_0^1$$

$$= \boxed{-\pi}$$

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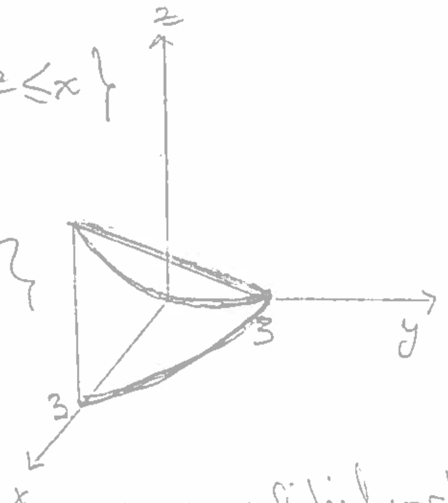
1. Consider the solid **in the first octant** that lies within the cylinder $x^2 + y^2 = 9$ below the plane $z = x$. This solid has a mass density given by the function $\delta(x, y, z) = y$. Find the total mass of this solid.

Here $E = \{(x,y,z) \mid x^2+y^2 \leq 9,\ x \geq 0,\ y \geq 0,\ 0 \leq z \leq x\}$



In cylindrical coords

$E = \{(r,\theta,z) \mid 0 \leq r \leq 3,\ 0 \leq \theta \leq \pi/2,\ 0 \leq z \leq r\cos\theta\}$

&

$\delta(r,\theta,z) = r\sin\theta$.

So, the mass is

$$m = \iiint_E y\, dV = \int_0^3 \int_0^{\pi/2} \int_0^{r\cos\theta} r\sin\theta \underbrace{r\, dz\, d\theta\, dr}_{dV \text{ in cylindrical coord}}$$

$$= \int_0^3 \int_0^{\pi/2} r^3 \cos\theta \sin\theta \, d\theta\, dr$$

$$= \int_0^3 \frac{r^3}{2} \sin^2\theta \Big|_0^{\pi/2} dr$$

$$= \int_0^3 \frac{r^4}{8} dr$$

$$= \boxed{\frac{81}{8}}$$

P.S.: One may use: $E = \{(x,y,z) \mid 0 \leq x \leq 3,\ 0 \leq y \leq \sqrt{9-x^2},\ 0 \leq z \leq x\}$

However, you'd need to convert to polar coords to evaluate the double integral.

2. Consider the vector field $\vec{F} = \langle y, xy \rangle$, and let C be the upper half-circle $x^2 + y^2 = 1$ oriented from (1,0) to (-1,0). Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r}.$$

• Parametrization: $\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$, $0 \leq \theta \leq \pi$.

so that $\vec{r}(\theta) = \langle \cos\theta, \sin\theta \rangle$, $0 \leq \theta \leq \pi$

$\Rightarrow \vec{r}\,'(\theta) = \langle -\sin\theta, \cos\theta \rangle$,

& $\vec{F}(\vec{r}(\theta)) = \langle \sin\theta, \cos\theta \sin\theta \rangle$

• Therefore

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^\pi \langle \sin\theta, \cos\theta\sin\theta \rangle \cdot \langle -\sin\theta, \cos\theta \rangle \, d\theta$$

$$= \int_0^\pi (-\sin^2\theta + \cos^2\theta \sin\theta)\, d\theta$$

$$= \int_0^\pi \left(-\left(\frac{1-\cos(2\theta)}{2}\right) + \cos^2\theta \sin\theta \right) d\theta$$

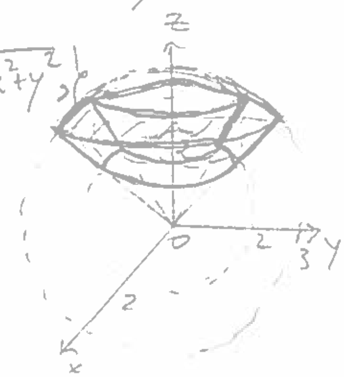
$$= \left(-\left(\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right) - \frac{\cos^3\theta}{3} \right) \Big|_0^\pi$$

$$= \boxed{-\frac{\pi}{2} + \frac{2}{3}}$$

3. Consider the solid **in the first octant** which lies in between the cones $z = \sqrt{3}\sqrt{x^2+y^2}$ and $z = \sqrt{x^2+y^2}$, and the spheres $x^2+y^2+z^2 = 4$, and $x^2+y^2+z^2 = 9$. This solid has a mass density given by the function $\delta(x, y, z) = z^2$. Set up a triple integral in **spherical coordinates** which gives the total mass of this solid. **DO NOT EVALUATE THE INTEGRAL.**

(FYI: the angles that make the cones $z = \sqrt{3}\sqrt{x^2+y^2}$ & $z = \sqrt{x^2+y^2}$ with the positive z-axis are respectively $\frac{\pi}{6}$ & $\frac{\pi}{3}$.)

Here $E = \{(x,y,z) \mid 4 \leq x^2+y^2+z^2 \leq 9,\ \sqrt{x^2+y^2} \leq z \leq \sqrt{3}\sqrt{x^2+y^2},$
$x \geq 0,\ y \geq 0,\ z \geq 0\}$



So, in spherical coords:

$$E = \{(\rho, \theta, \phi) \mid 2 \leq \rho \leq 3,\ 0 \leq \theta \leq \frac{\pi}{2},\ \frac{\pi}{6} \leq \phi \leq \frac{\pi}{4}\}$$

& $\delta(\rho, \theta, \phi) = \rho^2 \cos^2\phi$.

Therefore, the total mass is

$$\boxed{m = \int_2^3 \int_0^{\pi/2} \int_{\pi/6}^{\pi/4} \rho^2 \cos^2\phi \ \overbrace{\rho^2 \sin\phi \, d\phi \, d\theta \, d\rho}^{dV \text{ in spherical coords}}}$$

4. Find the total arclength of the curve which is parametrized by the following vector function

$$\vec{r}(t) = \sin(3t)\,\vec{\mathbf{i}} + \cos(3t)\,\vec{\mathbf{j}} + (3t+1)\,\vec{\mathbf{k}}, \quad 0 \le t \le \pi/9.$$

$$\Longrightarrow \vec{r}\,'(t) = \langle 3\cos(3t), -3\sin(3t), 3 \rangle$$

$$\& \ \|\vec{r}\,'(t)\| = \sqrt{9\cos^2(3t) + 9\sin^2(3t) + 9}$$

$$= \sqrt{18} = 3\sqrt{2}$$

$$\therefore \ L = \int_0^{\pi/9} \|\vec{r}\,'(t)\|\, dt = \boxed{\frac{\pi\sqrt{2}}{3}}$$

5. *Part (a) is independent of parts (b) and (c).*

(a) Determine which among the following vector fields is a conservative vector field. Justify your answer.

$$\vec{F} = x^2 \cos y\vec{\mathbf{i}} - 2x\sin y\vec{\mathbf{j}}$$
$$\vec{G} = 2xe^{-y}\vec{\mathbf{i}} + (2y - x^2e^{-y})\vec{\mathbf{j}} + 2z\vec{\mathbf{k}}$$
$$\vec{H} = \cos(y)\vec{\mathbf{i}} - (x\sin(y) + y)\vec{\mathbf{j}}$$
$$\vec{K} = \cos(xz)\vec{\mathbf{i}} + \sin(yz)\vec{\mathbf{j}} + xy\sin(z)\vec{\mathbf{k}}$$

(b) Find a potential function f for the conservative vector field $\vec{F} = y\cos(x)\vec{\mathbf{i}} + \sin(x)\vec{\mathbf{j}}$.

(c) Evaluate the line integral of **the conservative vector field** in (b) along the curve C, where C is any smooth curve from $(0,0)$ to $(\pi, 1)$.

(a) • For $\vec{F}$, $P(x,y) = x^2\cos y$ & $Q(x,y) = -2x\sin y$.

So, $P_y = -x^2\sin y \neq Q_x = -2\sin y$ & thus $\vec{F}$ is not conservative

• For $\vec{H}$, $P(x,y) = \cos y$ & $Q(x,y) = -x\sin y - y$

So, $P_y = -\sin y = Q_x$ & thus $\vec{H}$ is conservative.

• For $\vec{G}$, we have $\text{Curl}\,\vec{G} = \nabla \times \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xe^{-y} & 2y - x^2e^{-y} & 2z \end{vmatrix}$

$= \langle 0, 0, 0 \rangle$

& thus $\vec{G}$ is conservative.

• For $\vec{K}$, we have $\text{Curl}\,\vec{K} = \nabla \times \vec{K} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \cos(xz) & \sin(yz) & xy\sin z \end{vmatrix}$

$= \langle x\sin z - y\cos z, -x\sin(xz) - y\sin z, 0 \rangle$
$\neq \hat{0}$

& so $\vec{K}$ is not conservative.

Extra page for Question 5. v.2

(b) Let $f(x,y)$ be such that $\vec{F} = \nabla f \iff \begin{cases} f_x = y\cos x \\ f_y = \sin x. \end{cases}$

Then $f(x,y) = \int y \cos x \, dx \quad \left(\overset{\text{or}}{=} \int \sin x \, dy \right)$

$= y \sin x + g(y) \quad \left(\overset{\text{or}}{=} y \sin x + h(x) \right).$

But $f_y = \sin x \iff \sin x + \frac{d}{dy} g(y) = \sin x$

$\iff \frac{d}{dy} g(y) = 0$

$\implies g(y) = K, \; K \in \mathbb{R}$

Take $K = 0$.

(or $f_x = y \cos x$

$\iff y \sin x + \frac{d}{dx} h(x) = y \cos x$

$\iff \frac{d}{dx} h(x) = 0$

$\implies h(x) = K, \; K \in \mathbb{R}$

Take $K = 0$.

$\therefore \quad \boxed{f(x,y) = y \sin x}$

6. Use Green's Theorem to evaluate the line integral $\oint_C xy^2\,dx + x\,dy$, where C is the unit circle with counterclockwise (CCW) orientation (which positive orientation).

Here $\vec{F} = \langle xy^2 = P(x,y),\ x = Q(x,y) \rangle$.

So, $Q_x - P_y = 1 - 2xy$ $(= 1 - 2r^2\cos\theta\sin\theta$ in polar coords$)$

Also, since C is the unit circle (with CCW orientation),

D is the unit disk. $D = \{(x,y) \mid x^2+y^2 \le 1\}$
$(= \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$ in polar coords.

By Green's Thm, we have

$$\oint_C xy^2\,dx + x\,dy = \iint_D (Q_x - P_y)\,dA$$

$$= \iint_D (1 - 2xy)\,dA$$

(In polar coords)

$$= \int_0^1 \int_0^{2\pi} (1 - 2r^2\cos\theta\sin\theta)\,\overbrace{r\,d\theta\,dr}^{dA \text{ in}}$$

$$= \int_0^1 r(\theta - r^2\sin^2\theta)\Big|_0^{2\pi}\,dr$$

$$= \int_0^1 2\pi r\,dr$$

$$= \pi r^2\Big|_0^1$$

$$= \boxed{\pi}$$