University of Ottawa
Department of Mathematics and Statistics
Calculus III for Engineers
MAT 2322 X00 - Spring-Summer 2019
Midterm I - V.1
Professor: Abdelkrim El basraoui
June 11, 2019

Name:______________________________ ID Number:____________________

Instructions: (Please read carefully.)

- This exam has 8 pages and 6 questions, and you have 80 minutes to complete it.
- This is a closed book exam.
- **The only calculators which are allowed are those approved by the faculty of science such as Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
- Questions 1, 2, 4, 5 and 6 are worth 7 marks each, and question 3 is worth 5 marks, so organize your time accordingly.
- Answer each question in the space provided, using backs of pages or the extra page at the end if necessary.
- **A correct answer requires a full, clearly-written and detailed solution.**
- Do not unstaple the test.
- Good luck!
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Signature: ____________________________________

Question	Q.1	Q.2	Q.3	Q.4	Q.5	Q.6	Total
Maximum	7	7	5	7	7	7	40
Score							

1. Find and classify the critical points of the function $f(x,y) = 4x - 3x^3 - 2xy^2$.

• Critical pts: $\nabla f = <\underbrace{4-9x^2-2y^2}_{f_x}, \underbrace{-4xy}_{f_y}> = <0,0>$

$\iff \begin{cases} 4-9x^2-2y^2=0 \\ -4xy=0 \end{cases} \iff \begin{cases} 4-9x^2-2y^2=0 & \text{Eq.1} \\ x=0 \text{ or } y=0 \end{cases}$

If $x=0$, Eq.1 implies that $4-2y^2=0 \iff y=\pm\sqrt{2}$ & we have the critical points $(0,\pm\sqrt{2})$.

If $y=0$, Eq.1 implies that $4-9x^2=0 \iff x=\pm\frac{2}{3}$ & we get the critical points $(\pm\frac{2}{3},0)$.

• Classification: $f_{xx}=-18x$; $f_{xy}=f_{yx}=-4y$; $f_{yy}=-4x$

$\implies D(x,y) = 72x^2 - 16y^2$.

Therefore, we have

- $D(0,\pm\sqrt{2}) = -32 < 0 \implies (0,\pm\sqrt{2})$ are saddle points.
- $D(-2/3, 0) = 32 > 0$ & $f_{xx}(-\frac{2}{3},0) = 12 > 0 \implies (-\frac{2}{3},0)$ is a local minimum.
- $D(\frac{2}{3},0) = 32 > 0$ & $f_{xx}(\frac{2}{3},0) = -12 < 0 \implies (\frac{2}{3},0)$ is a local maximum.

2. Use Lagrange Multipliers to find the maximum and minimum values of the function $f(x,y) = xy^2$ on the ellipse $4x^2 + 9y^2 = 36$. $= g(x,y) \leftarrow$ constraint.

• We solve $\nabla f = \lambda \nabla g$ (& $g(x,y) = 36$).

$\nabla f = \langle y^2, 2xy \rangle = \lambda \langle 8x, 18y \rangle$.

$\iff \begin{cases} y^2 = 8\lambda x \\ 2xy = 18\lambda y \end{cases} \iff \begin{cases} y^2 = 8\lambda x \\ 2y(x - 9\lambda) = 0 \end{cases} \iff \begin{cases} y^2 = 8\lambda x & \text{Eq.1} \\ y = 0 \text{ or } \lambda = \frac{1}{9}x \end{cases}$

If $y = 0$, the constraint gives $x = \pm 3$ & we get the pts $(\pm 3, 0)$.

If $\lambda = \frac{1}{9}x$, Eq.1 gives $y^2 = \frac{8}{9}x^2$ & the constraint implies that

$4x^2 + 9\left(\frac{8}{9}x^2\right) = 12x^2 = 36 \implies x = \pm\sqrt{3}$.

But $y^2 = \frac{8}{9}x^2 \implies y = \pm\frac{2\sqrt{2}}{3}x$. So, if $x = -\sqrt{3}$, we get the pts $\left(-\sqrt{3}, \pm\frac{2\sqrt{6}}{3}\right)$

& if $x = \sqrt{3}$, we get the pts $\left(\sqrt{3}, \pm\frac{2\sqrt{6}}{3}\right)$.

• Conclusion: $f(\pm 3, 0) = 0$.

$f\left(-\sqrt{3}, \pm\frac{2\sqrt{6}}{3}\right) = -\frac{8\sqrt{3}}{3}$.

$f\left(\sqrt{3}, \pm\frac{2\sqrt{6}}{3}\right) = \frac{8\sqrt{3}}{3}$.

$\implies$ $-\frac{8\sqrt{3}}{3}$ is the min. value of f (at the pts $\left(-\sqrt{3}, \pm\frac{2\sqrt{6}}{3}\right)$),

& $\frac{8\sqrt{3}}{3}$ is the max. value of f (at the pts $\left(\sqrt{3}, \pm\frac{2\sqrt{6}}{3}\right)$).

Bottom surface

3. What is the volume of the solid between the planes $z = x + y$ and $z = 5x + 3y$ and lying above the rectangle $R = [0,1] \times [0,2]$?

Top surface

$$V = \iint_R (\text{Top Surface} - \text{Bottom Surface})\, dA$$

$$V = \iint_R \left((5x+3y)-(x+y)\right) dA \quad \left(\text{or} = \iint_R \left(\int_{x+y}^{5x+3y} 1\, dz\right) dA \right)$$

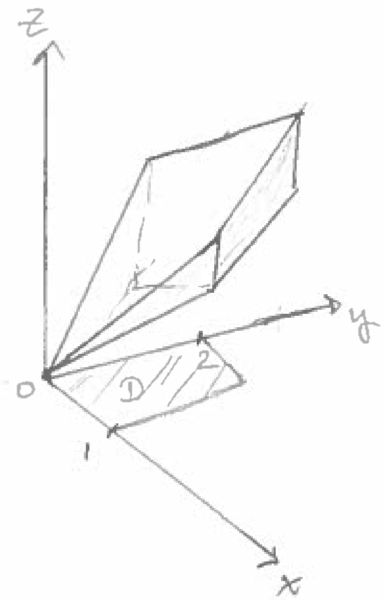
$$= \int_0^1 \int_0^2 (4x+2y)\, dy\, dx \quad \left(\overset{\text{or}}{=} \int_0^2 \int_0^1 (4x+2y)\, dx\, dy \,.\right)$$

$$= \int_0^1 \left(4xy+y^2\right)\Big|_0^2\, dx$$

$$= \int_0^1 (8x+4)\, dx$$

$$= 4x^2+4x \Big|_0^1$$

$$= \boxed{8}$$



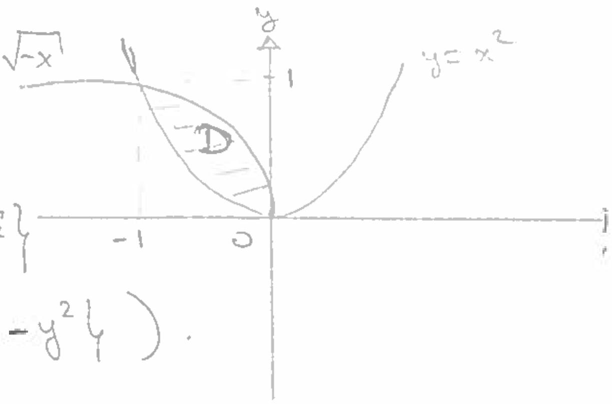
4. Let D be the region in the xy-plane bounded by the parabolas $y = x^2$ and $y = \sqrt{-x}$. **Sketch** the region D then **evaluate** the following double integral $\iint_D 6xy\, dA$.

(Intersection points: $x^2 = \sqrt{-x}$

$\iff x^4 = -x \iff x(x^3+1) = 0$

$\iff x = 0$ or $x^3 = -1$

$\iff x = 0$ or $x = -1$)



We have $D = \{(x,y) \mid -1 \le x \le 0,\ x^2 \le y \le \sqrt{-x}\}$

$\left(\text{or} = \{(x,y) \mid 0 \le y \le 1,\ -\sqrt{y} \le x \le -y^2\}\right).$

Therefore

$$\iint_D 6xy\, dA = \int_{-1}^{0}\int_{x^2}^{\sqrt{-x}} 6xy\, dy\, dx$$

$$= \int_{-1}^{0} \left(3xy^2\right)\Big|_{x^2}^{\sqrt{-x}} dx$$

$$= \int_{-1}^{0} \left(-3x^2 - 3x^5\right) dx$$

$$= \left(-x^3 - \frac{x^6}{2}\right)\Big|_{-1}^{0}$$

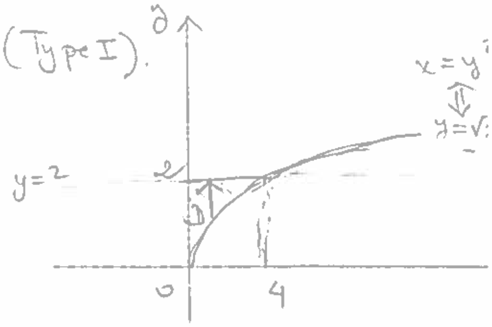
$$= \boxed{\frac{1}{2}}$$

5. Sketch the region of integration then **evaluate** the following double integral $\int_0^4 \int_{\sqrt{x}}^2 \cos(y^3)\, dy\, dx$.

Hint: use the region to reverse the order of integration.

We have $D = \{(x,y) \mid 0 \le x \le 4,\ \sqrt{x} \le y \le 2\}$ (Type I).

As a type II, $D = \{(x,y) \mid 0 \le y \le 2,\ 0 \le x \le y^2\}$.



Therefore

$$\int_0^4 \int_{\sqrt{x}}^2 \cos(y^3)\, dy\, dx = \int_0^2 \int_0^{y^2} \cos(y^3)\, dx\, dy$$

$$= \int_0^2 \left(x \cos(y^3)\right)\Big|_0^{y^2} dy$$

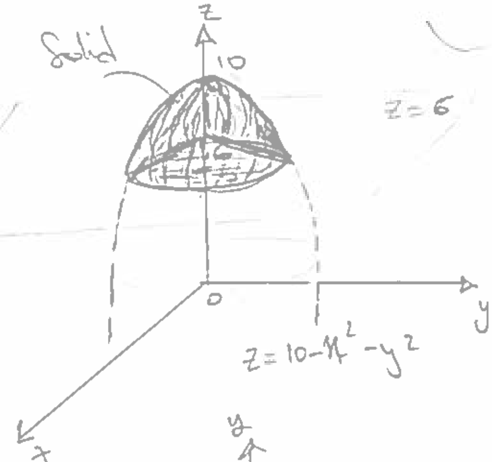
$$= \int_0^2 y^2 \cos(y^3)\, dy$$

Use a u-substitution ($u = y^3$)

$$= \frac{1}{3} \sin(y^3)\Big|_0^2$$

$$= \boxed{\frac{1}{3} \sin(8)}$$

6. Set up a double integral in **polar coordinates** to compute the volume of the solid in the first octant bounded from above by the paraboloid $z = 10 - x^2 - y^2$ and from below by the plane $z = 6$, then **evaluate it**.



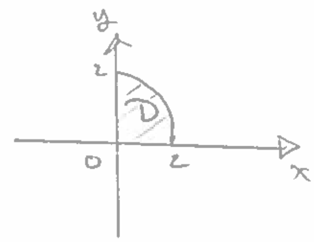
- Intersection: $10 - x^2 - y^2 = 6 \iff x^2 + y^2 = 4$.

Therefore $D = \{(x,y) \mid x^2+y^2 \le 4, \underbrace{x \ge 0, y \ge 0}\}$

because of 1st octant.

- In polar coords:

$D = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le \pi/2\}$.



- Volume: $V = \iint_D ((\text{Top Surface}) - (\text{Bottom surface}))\, dA$

$= \iint_D ((10 - x^2 - y^2) - (6))\ dA$

$= \iint_D (4 - x^2 - y^2)\, dA$

Now, in polar coords: $4 - x^2 - y^2 = 4 - r^2$, $dA = r\,dr\,d\theta \ (= r\,d\theta\,dr)$

Thus $V = \int_0^{\pi/2}\int_0^2 (4 - r^2)\, r\,dr\,d\theta \quad \left(\overset{\text{or}}{=} \int_0^2\int_0^{\pi/2} (4-r^2)\, r\,d\theta\,dr\right)$

$= \int_0^{\pi/2} \left(2r^2 - \frac{r^4}{4}\right)\Big|_0^2\ d\theta$

$= \int_0^{\pi/2} 4\ d\theta$

$= \boxed{2\pi}$

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1. Find and classify the critical points of the function $f(x,y) = 4y - 3y^3 - 2x^2y$.

- Critical points: $\nabla f = \langle \overbrace{-4xy}^{f_x}, \overbrace{4-9y^2-2x^2}^{f_y} \rangle = \langle 0, 0 \rangle$

$$\Longleftrightarrow \begin{cases} -4xy = 0 \\ 4-9y^2-2x^2 = 0 \end{cases} \Longleftrightarrow \begin{cases} x=0 \text{ or } y=0 \\ 4-9y^2-2x^2=0 \quad \text{Eq. 2} \end{cases}$$

If $x=0$, Eq. 2 gives $4-9y^2=0 \Longleftrightarrow y = \pm\frac{2}{3}$ & we get the critical pts $\underline{(0, \pm 2/3)}$.

If $y=0$, Eq. 2 implies that $4-2x^2=0 \Longleftrightarrow x = \pm\sqrt{2}$ & we get the critical pts $\underline{(\pm\sqrt{2}, 0)}$.

- Classification: $f_{xx} = -4y$; $f_{xy} = f_{yx} = -4x$; $f_{yy} = -18y$.

$\Longrightarrow D(x,y) = 72y^2 - 16x^2$.

Therefore, we have

- $D(\pm\sqrt{2}, 0) = -32 < 0 \Longrightarrow (\pm\sqrt{2}, 0)$ are saddle pts.
- $D(0, -2/3) = 32 > 0$ & $f_{xx}(0, -\frac{2}{3}) = \frac{8}{3} > 0 \Longrightarrow (0, -\frac{2}{3})$ is a local minimum.
- $D(0, \frac{2}{3}) = 32 > 0$ & $f_{xx}(0, \frac{2}{3}) = \frac{8}{3} < 0 \Longrightarrow (0, \frac{2}{3})$ is a local maximum.

2. Use Lagrange Multipliers to find the maximum and minimum values of the function $f(x,y) = x^2y$ on the ellipse $\underbrace{9x^2+4y^2}=36$. $= g(x,y) \leftarrow$ constraint.

- We solve $\nabla f = \lambda \nabla g$ ($\&$ $g(x,y)=36$).

$\nabla f = \langle 2xy, x^2 \rangle = \lambda \langle 18x, 8y \rangle$

$\Leftrightarrow \begin{cases} 2xy = 18\lambda x \\ x^2 = 8\lambda y \end{cases} \Leftrightarrow \begin{cases} 2x(y-9\lambda)=0 \\ x^2 = 8\lambda y \end{cases} \Leftrightarrow \begin{cases} x=0 \text{ or } \lambda = \frac{1}{9}y. \\ x^2 = 8\lambda y. \quad \text{Eq. 2} \end{cases}$

If $x=0$, the constraint gives $y=\pm 3$ & we get the pts $\underline{(0,\pm 3)}$.

If $\lambda = \frac{1}{9}y$, then Eq. 2 gives $x^2 = \frac{8}{9}y^2$ & the constraint implies that $9\left(\frac{8}{9}y^2\right)+4y^2 = 12y^2 = 36 \Rightarrow y = \pm\sqrt{3}$. But, $x^2 = \frac{8}{9}y^2$ $\Rightarrow x = \pm\frac{2\sqrt{2}}{3}y$. So, if $y=-\sqrt{3}$, $\Rightarrow x = \pm\frac{2\sqrt{6}}{3}$ & we get the pts $\underline{\left(\pm\frac{2\sqrt{6}}{3}, -\sqrt{3}\right)}$;

& if $y=\sqrt{3} \Rightarrow x = \pm\frac{2\sqrt{6}}{3}$ & we get the pts $\underline{\left(\pm\frac{2\sqrt{6}}{3}, \sqrt{3}\right)}$.

- Conclusion: $f(0,\pm 3) = 0$.

$f\left(\pm\frac{2\sqrt{6}}{3}, -\sqrt{3}\right) = -\frac{8\sqrt{3}}{3}$.

$f\left(\pm\frac{2\sqrt{6}}{3}, \sqrt{3}\right) = \frac{8\sqrt{3}}{3}$.

$\Rightarrow -\frac{8\sqrt{3}}{3}$ is the min. value of f (at the pts $\left(\pm\frac{2\sqrt{6}}{3}, -\sqrt{3}\right)$), & $\frac{8\sqrt{3}}{3}$ is the max. value of f (at the pts $\left(\pm\frac{2\sqrt{6}}{3}, \sqrt{3}\right)$).

Bottom surface

3. What is the volume of the solid between the planes $z = x + y$ and $z = 3x + 5y$ and lying above the rectangle $R = [0, 2] \times [0, 1]$?

Top surface.

$$V = \iint_R (\text{Top surface} - \text{Bottom surface})\, dA.$$

$$V = \iint_R ((3x + 5y) - (x+y))\, dA \quad \left(\overset{\text{or}}{=} \iint_R \left(\int_{x+y}^{3x+5y} 1\, dz \right) dA \right)$$

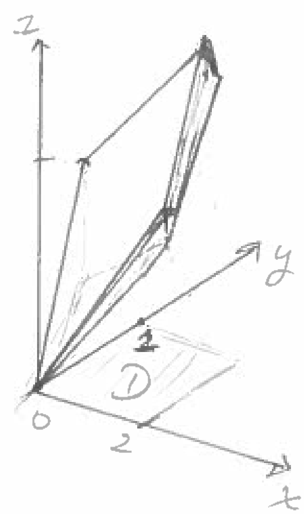
$$= \int_0^2 \int_0^1 (2x + 4y)\, dy\, dx \quad \left(\overset{\text{or}}{=} \int_0^1 \int_0^2 (2x+4y)\, dx\, dy \right)$$

$$= \int_0^2 (2xy + 2y^2) \Big|_0^1 dx$$

$$= \int_0^2 (2x + 2)\, dx$$

$$= x^2 + 2x \Big|_0^2$$

$$= \boxed{8}$$

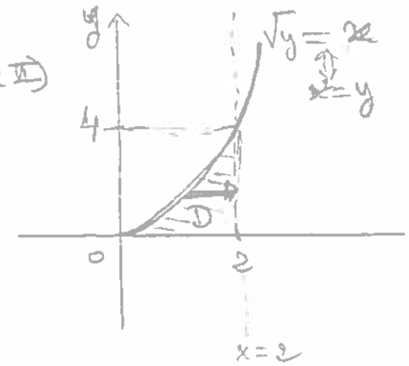


4. Let D be the region in the xy-plane bounded by the parabolas $y = x^2$ and $y =$ √[redacted]x
Sketch the region D then **evaluate** the following double integral $\iint_D 6xy\, dA$.

See Version 1

5. Sketch the region of integration then **evaluate** the following double integral $\int_0^4 \int_{\sqrt{y}}^2 \sin(x^3)\, dx\, dy$.

Hint: use the region to reverse the order of integration.



We have $D = \{(x,y) \mid 0 \le y \le 4,\ \sqrt{y} \le x \le 2\}$ (Type II)

As a type I, $D = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le x^2\}$.

Thus

$$\int_0^4 \int_{\sqrt{y}}^2 \sin(x^3)\, dx\, dy = \int_0^2 \int_0^{x^2} \sin(x^3)\, dy\, dx$$

$$= \int_0^2 \left(y \sin(x^3)\right)\Big|_0^{x^2} dx$$

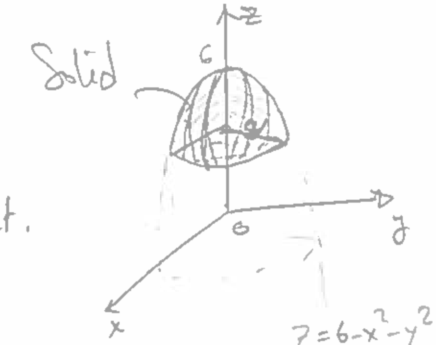
$$= \int_0^2 x^2 \sin(x^3)\, dx$$

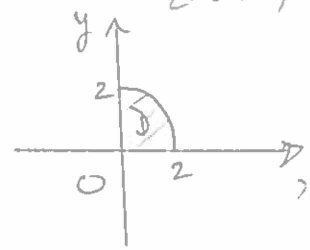
use a U-substitution ($u = x^3$)

$$= -\frac{1}{3}\cos(x^3)\Big|_0^2$$

$$= \boxed{-\frac{1}{3}(\cos(8) - 1)}$$

6. Set up a double integral in **polar coordinates** to compute the volume of the solid <u>in the first octant</u> bounded from above by the paraboloid $z = 6 - x^2 - y^2$ and from below by the plane $z = 2$, then **evaluate it.**





• Intersection: $6-x^2-y^2 = 2 \iff x^2+y^2=4$

Therefore $D=\{(x,y) \mid x^2+y^2 \leq 4, \underbrace{x \geq 0, y \geq 0}_{\text{because of } 1^{st} \text{ octant.}}\}$

• In polar coords

$$D=\{(r,\theta) \mid 0 \leq r \leq 2,\ 0 \leq \theta \leq \pi/2\}.$$

• Volume: $V = \iint_D ((\text{Top surface}) - (\text{Bottom surface}))\, dA$

$$= \iint_D ((6-x^2-y^2) - (2))\, dA$$

$$= \iint_D (4-x^2-y^2)\, dA$$

Now, in polar coords $4-x^2-y^2 = 4-r^2$; $dA = r\,dr\,d\theta \,(= r\,d\theta\,dr)$

Thus $V = \int_0^{\pi/2}\int_0^2 (4-r^2)\, r\,dr\,d\theta \quad \left(\overset{\text{or}}{=} \int_0^2\int_0^{\pi/2}(4-r^2)\, r\,d\theta\,dr\right)$

$$= \int_0^{\pi/2} \left(2r^2 - \frac{r^4}{4}\right)\Big|_0^2 d\theta$$

$$= \int_0^{\pi/2} 4\, d\theta$$

$$= \boxed{2\pi}$$