Concordia University   Department of CSE
COMP 232   Mathematics for Computer Science

Handout on invertibility of linear functions

1 Linear functions from R^2 to R^2

A function $f : R^2 \to R^2$ is called **linear** if it can be written as

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

where $a_{ij} \in R$ for $i, j \in \{1, 2\}$. Such a function is invertible if the determinant

$$D = a_{11}a_{22} - a_{12}a_{21} \neq 0$$

.

The inverse of the function f is given by:

$$f^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{D}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Example 1: Let $f(m, n) = (3m + 2n, 4m + 3n)$. Is f invertible?

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The determinant is $9 - 8 = 1$. Therefore it is invertible. The inverse function is:

$$f^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Example 2. Let $f(x, y) = (x - y, y - x)$. Is f invertible?

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

It is easy to see that the determinant $D = 1 - 1 = 0$. Therefore the function is not invertible.

2 Linear functions from Z^2 to Z^2

The same function $f : Z^2 \to Z^2$ is called linear if it can be written as

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where $a_{ij} \in Z$ for $i, j \in \{1, 2\}$. Such a function is invertible if the following two conditions are met.

1. the determinant
$$D = a_{11}a_{22} - a_{12}a_{21} \neq 0$$
2. D divides a_{ij} for $i, j \in \{1, 2\}$

Example 3. It is easy to see that the function in Example 1 is also invertible when considered from Z^2 to Z^2 since $D = 1$ and divides all entries of the matrix.

Example 4. Let $f(x, y) = (3x + y, 4x + 3y)$. Is f invertible?

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The determinant $D = 9 - 4 = 5 \neq 0$. However, since 5 does not divide the entries of the matrix, the function is not invertible.