Solution to Test 1 (version A)
MAT1320C, Fall 2012
Total = 20 marks

1. (3 marks) The following is a table of some values of two functions $y = f(x)$ and $y = g(x)$.

x	1	2	3	4
$f(x)$	4	3	2	1
$g(x)$	2	4	3	1

Find (i) $(f \circ g)(2)$, (ii) $(g \circ f)(2)$, and (iii) $(g \circ g)(1)$.

Solution. $(f \circ g)(2) = f(g(2)) = 1$. $(g \circ f)(2) = g(f(2)) = 3$. $(g \circ g)(1) = g(g(1)) = 4$.

2. (4 marks) Consider the function $f(x) = \dfrac{\ln x}{1 - \ln x}$.

(i) (1 mark) What is the domain of function *f* ?

(ii) (2 marks) Find the inverse of function *f*.

(iii) (1 marks) What is the range of function *f* ? (The question asks the range of function *f*, not the range of the inverse of *f*).

Solution. (i) $x > 0$ and $x \neq e$.

(ii) $y - y \ln x = \ln x$. $\ln x\,(1 + y) = y$. $\ln x = \dfrac{y}{1+y}$, $x = e^{\frac{y}{1+y}}$. The inverse function is $y = e^{\frac{x}{1+x}}$.

(iii) The range of function *f* is the domain of f^{-1}, which is all real numbers except -1.

3. (3 marks) Exponential and Logarithmic functions:

(1) (1 mark) What is $\log_2 \dfrac{1}{\sqrt{2}}$?

Solution. Since $\sqrt{2} = 2^{-1/2}$, $\log_2 \dfrac{1}{\sqrt{2}} = -\dfrac{1}{2}$.

(2) (2 marks) Solve for *x*, if $\log_2(3x + 2) - \log_2(x - 1) = 0$.

Solution. $\log_2 \dfrac{3x+2}{x-1} = 0, \dfrac{3x+2}{x-1} = 1$. $3x + 2 = x - 1$. $2x = -3$, $x = -3/2$.

4. (4 marks) Finding limits:

(1) $\lim_{x\to 0}\left(\frac{1-\sqrt{1+x}}{x}\right)=-\frac{1}{2}$.

(2) $\lim_{x\to\infty}\frac{3x^2-1}{x^2+2x}=3$.

Solution. (i)

$$\lim_{x\to 0}\left(\frac{1-\sqrt{1+x}}{x}\right)=\lim_{x\to 0}\left(\frac{(1-\sqrt{1+x})(1+\sqrt{1+x})}{x(1+\sqrt{1+x})}\right)=\lim_{x\to 0}\left(\frac{1-(1+x)}{x(1+\sqrt{1+x})}\right)=\lim_{x\to 0}\left(\frac{-x}{x(1+\sqrt{1+x})}\right)$$

$$=-\lim_{x\to 0}\left(\frac{1}{1+\sqrt{1+x}}\right)=-\frac{1}{2}.$$

(ii) $\lim_{x\to\infty}\frac{3x^2-1}{x^2+2x}=\lim_{x\to\infty}\frac{3-1/x^2}{1+2/x}=3$.

5. (2 marks) **Use the definition of the derivative** to find the derivative of the functions $y = x^{-2}$ when $x = 1$.

Solution. $y'(1) = -2$.

$y(1 + h) = (1 + h)^{-2}$, $y(1) = 1$,

$$\lim_{h\to 0}\frac{y(1+h)-y(1)}{h}=\lim_{h\to 0}\frac{(1+h)^{-2}-1}{h}=\lim_{h\to 0}\frac{1-(1+h)^2}{h(1+h)^2}=\lim_{h\to 0}\frac{-2h-h^2}{h(1+h)^2}=\lim_{h\to 0}\frac{-2-h}{(1+h)^2}=-2.$$

6. (4 marks) Consider a continuous function $y = f(x)$ defined for all real values of x. Suppose this function satisfies all of the following conditions:

(a) $f(3) = 2$, $f(0) = 5$;
(b) $f'(x) < 0$ when $x > 0$, $f'(x) > 0$ when $x < 0$.
(c) $f''(x) < 0$ when $x < 2$, $f''(x) > 0$ when $x > 2$.
(d) $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = -\infty$.

Sketch the graph of this function.

Solution.