COMP. 233.

ASSIGNMENT 1.

1. Alice has a magic die which has the property that when rolled, the probability of getting a number is proportional to the number. That is, there is some constant *c* such that the probability of getting the number *i* when the die is rolled is *ci*.

   A. What is the value of *c*?
   B. What is the probability that Alice rolls a 5 or a 6?
   C. What is the probability that Alice gets an odd number when she rolls the die?

2. Prove for any three events *A*, *B*, *C*:
   $P(ABC) \geq P(A) + P(B) + P(C) - 2$.

3. Bob has a pack of well-shuffled 52 cards, and keeps picking cards out
   at random until he draws the Ace of spades. What is the probability
   that the Ace of Spades is the 25th. card to be pulled out if:

   A. Bob discards each of the first 24 cards after he pulls it out?
   B. Bob replaces each card he pulls out if it is not the Ace of Spades?

4. Suppose we throw 5 labelled balls randomly into 3 labelled bins.

   A. How many outcomes are there in the sample space?
   B. What is the probability that the first bin is empty?
   C. What is the probability that at least two bins are empty?
   D. What is the probability that no bin is empty?
   E. What is the probability that the first bin is empty, given that the second bin is empty?

5. Suppose we throw 11 indistinguishable balls randomly into 3 labeled bins.

   A. How many outcomes are there in the sample space?
   B. What is the probability that the first bin is empty?
   C. What is the probability that no bin is empty?

6. A hash table is a data structure that supports the storage of subset of keys from a very large set *S*. To add a key *x* to a hash table, we use the hash function *h*: we compute *h*(*x*) and store *x* at location *h*(*x*). If two values *x* and *y* hash to the same location, we say we have a *collision.* For the hash table to work efficiently, we want to minimize the probability of collisions. The hash function is uniform in the sense that each key is equally likely to be hashed to any location in the table. Suppose we want to store 20 distinct keys into a hash table of size 100. What is the probability that there is no collision?

7. At an international 100 m race, there are 4 Americans, 2 Jamaicans, 2 Canadians, 1 Russian, and 1 Chinese runner competing.

   A. What are the total number of possible outcomes for the race?
   B. If the scoring only considers the country and not the individual identities of the runners, how many outcomes are possible?
   C. Assuming all possible outcomes for the race are equally likely, what is the probability the first three places are won by runners from 3 different countries?

8. A bag contains the following blocks: 4 large red blocks, 6 large blue blocks, 6 small red blocks, and x small blue blocks. We draw a block at random from the bag. Let A be the event that the block is red and B be the event that the block is large. For what value of x are A and B independent events?

9. You ask your neighbor to water a sickly plant while you are away on vacation. Without water, the plant will die with probability 0.8, and with water, it will die with probability 0.2. You are 80% certain that your neighbor will water the plant.

   A. What is the probability that the plant will be dead when you return?
   B. If when you return, the plant is dead, what is the probability that your neighbor forgot to water the plant?

10. An insurance company classifies people into one of three classes: good risks, average risks, and bad risks. Their records indicate that over a one-year span, the probabilities that good, average, and bad risk people are involved in an accident are 0.05, 0.15, and 0.3 respectively. The company classifies 20% of the population as good risks, 50% as average risks, and 30% as bad risks.

    A. What proportion of people have accidents in a fixed year?
    B. If Alice had an accident last year, what is the probability that she was classified as a good risk?

11. How many bit strings of length 10 containing at least 5 consecutive 0's or at least 5 consecutive 1's are there?

12. A Palindrome is a string whose reversal is identical to the original string.
    How many bit strings of length N are Palindromes?

13. A Peruvian human sacrificial temple, near Lima, Peru, has three levels, levels J, M, and H, where tourists have the only access to the temple.

The probabilities that a tourist visiting the temple will visit the different levels are:

Visit level J: 0.74
Visit level M: 0.70
Visit level H: 0.62
Visit levels J and M: 0.52
Visit levels J and H: 0.46
Visit levels M and H: 0.44
Visit levels J and M and H: 0.34.

Find the probabilities that a person visiting the temple will:

A. Visit level M given that he will go to level J.
B. Visit level H given that he will go to level J and level M.
C. Not visit level J given that he will visit level M or visit level H.
D. Visit level H and visit level J given that he will not visit level M.

14. A company receives a shipment of 24 computers, of which 5 are defective. The Quality Control Department (Q.C.) samples 7 of these computers.

What is the probability that:

A. Q.C. finds 0 defectives in its sample of 7?
B. Q.C. finds exactly 2 defectives in its sample of 7?
C. Q.C. finds at least 1 defective computer in its sample of 7?

15. Ten parallel lines are perpendicular to ten other parallel lines. What is the total number of rectangles formed?

16. To transfer into a particular technical department, a company requires an employee to pass a screening test. A maximum of three attempts are allowed, at six-month intervals between trials. From past records, it is found that 40% pass on the first trial; of those that fail the first trial and take the test a second time, 60% pass; and of those that fail on the second trial and take the test a third time, 20% pass.
For an employee desiring to transfer:

A. What is the probability of passing the test on the first or second try?

B. What is the probability of failing on all three attempts?

C. What is the probability of failing on the first two trials, and passing on the third?

17. What is the probability that a number chosen at random from the first 1000 positive integers is exactly divisible by 24 or 36?

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COMP. 233 – Probability and Statistics for Computer Science.

Solution to Assignment 1

1. (a) Since $\sum_{i=1}^{6} ci = 1$ we obtain $c = 1/21$.
   (b)
   $$\frac{5}{21} + \frac{6}{21} = \frac{11}{21} \approx 0.524$$
   .
   (c)
   $$\frac{1}{21} + \frac{3}{21} + \frac{5}{21} = \frac{9}{21} = \frac{3}{7} \approx 0.429$$

2. We know that

$$\begin{aligned}
1 - P(ABC) &= P((ABC)^C) \\
&= P(A^C \cup B^C \cup C^C) \text{ using de Morgan's law} \\
&\leq P(A^C) + P(B^C) + P(C^C) \text{ since they are not necessarily disjoint} \\
&= (1 - P(A)) + (1 - P(B)) + (1 - P(C)) \\
&= 3 - P(A) - P(B) - P(C)
\end{aligned}$$

   Rearranging the terms we obtain the desired inequality.

3. (a) $\frac{51 \cdot 50 \cdots 28 \cdot 1}{52 \cdot 51 \cdots 28} = \frac{1}{52} \approx 0.019$
   (b) $\frac{51^{24}}{52^{24}} \times \frac{1}{52} \approx 0.012$

4. (a) $3^5 = 243$
   (b) Number of outcomes in which the first bin is empty is 2^5. So the desired probability is $\frac{2^5}{3^5} \approx 0.132$.
   (c) Since all three bins cannot be empty, the desired probability is just the probability that exactly two bins are empty. There are three ways of choosing these two bins, and since all the balls must go into the remaining bin, there are exactly three outcomes in which exactly two bins are empty. The desired probability is therefore $\frac{3}{3^5} \approx 0.012$.

(d) The solution to 4.D. is at the end of this document.

(e) Let A be the event that the first bin is empty, and B be the event that the second bin is empty.

$$P(A|B) = P(AB)/P(B) = \frac{1/3^5}{2^5/3^5} = \frac{1}{32} \approx 0.031$$

5. (a) The number of outcomes is equal to the number of non-negative solutions to the equation

$$x_1 + x_2 + x_3 = 11$$

which is $\binom{13}{2} = 78$.

(b) The number of outcomes in which the first bin is empty is equal to the number of non-negative solutions to

$$x_2 + x_3 = 11$$

which is $\binom{12}{1} = 12$

The desired probability is $\frac{12}{78} = \frac{2}{13} \approx 0.154$.

(c) The number of outcomes in which no bin is empty is equal to the number of *positive* solutions to the equation

$$x_1 + x_2 + x_3 = 11$$

Taking $\hat{x_i} = x_i - 1$, this is the same as the number of non-negative solutions to the equation

$$\hat{x_1} + \hat{x_2} + \hat{x_3} = 8$$

which is $\binom{10}{2} = 45$.

Thus the desired probability is $\frac{45}{78} = \frac{15}{26} \approx 0.577$

6. There are 100^{20} total outcomes for hashed locations for the 20 keys. There are $P_{100,20}$ outcomes in which every key is hashed to a different location (i.e. no collisions.). Therefore the probability that there is no collision is $\frac{P_{100,20}}{100^{20}} \approx 0.13$.

7. (a) $10! = 3628800$

(b)

$$\frac{10!}{4!2!2!1!1!} = 37800$$

(c) We now consider only *unordered* outcomes for the first three places, that is choices of the runners who placed first, second, or third, where the order between the three players does not matter. There are $\binom{10}{3} = 120$ outcomes in all.

Next we count the outcomes in which the first, second, and third places are won by runners from 3 different countries. There are 5 countries, and therefore $\binom{5}{3} = 10$

ways of choosing the three countries winning the first three places, however, the number of outcomes corresponding to all these are not the same. For example, if the winning countries are America, Jamaica, and Canada, there are 4 ways of choosing the American winner, 2 ways each of choosing the Jamaican and Canadian winners, which is a total of 16 outcomes. On the other hand, if the winning countries are China, Russia, and Canada, there are only 2 outcomes. Tabulating and adding all the outcomes, there are 64 outcomes in which the first, second, and third places are won by runners from 3 different countries. The desired probability is therefore $\frac{64}{120} = \frac{8}{15} \approx 0.533$.

8. *Solution:* $P(A) = 10/(16+x)$ and $P(B) = 10/(16+x)$. Also $P(AB) = 4/(16+x)$. For A and B to be independent, we require

$$P(AB) = P(A)P(B)$$

which gives: $(16+x) = 25$ which gives $x = 9$.

9. Let D be the event that the plant dies, and let W be the event that the plant is watered. We are told that $P(W) = 0.8$, $P(D|W) = 0.2$, and $P(D|W^C) = 0.8$.

(a)
$$P(D) = P(D|W)P(W) + P(D|W^C)P(W^C) = 0.2 \times 0.8 + 0.8 \times 0.2 = 0.32$$

(b)
$$P(W^C|D) = \frac{P(D|W^C)P(W^C)}{P(D)} = \frac{0.8 \times 0.2}{0.32} = 0.5$$

10. We denote the event that a person is good, average, and bad risk by $G.A, B$ respectively. We denote the event of having an accident by C.

(a) $P(C) = P(C|G)P(G) + P(C|A)P(A) + P(C|B)P(B) = 0.05 \times 0.2 + 0.15 \times 0.5 + 0.3 \times 0.3 = 0.175$

(b)
$$P(G|C) = \frac{P(C|G)P(G)}{P(C)} = \frac{0.01}{0.175} = 0.057$$

11. Let A: set containing all bit strings of length 10 with at least 5 consecutive 0's.

B: set containing all bit strings of length 10 with at least 5 consecutive 1's.

Set A. (Set B is similar).
The number of bit strings in set A is a function of where the 5 consecutive 0's start.
If the 5 consecutive 0's start:

At bit 1: 2^5 bits strings exist.
At bit 2: 2^4 bits strings exist.
At bit 3: 2^4 bits strings exist.
At bit 4: 2^4 bits strings exist.
At bit 5: 2^4 bits strings exist.
At bit 6: 2^4 bits strings exist.

$|A|$ = 112 bits strings exist in set A.
Because of symmetry $|B|$ = 112.
$\therefore$,

$$\text{Total number} = |A| + |B| - |A \cap B|$$
$$= 112 + 112 - 2$$
$$\text{Total number} = 222.$$

12. Let n: length of Palindrome.

If n is even.
If n is even then the $\frac{n}{2}$ leftmost (rightmost) positions will take on

$2^{n/2}$ values. The rightmost (leftmost) $\frac{n}{2}$ positions will take on the **same** values as the leftmost (rightmost) positions.

If n = 2: (2)(1) = 2 2 possible values.
If n = 4: (2)(2)(1)(1) = 4 4 possible values.
If n = 6: (2)(2)(2)(1)(1)(1) = 8 8 possible values.

If n even: $2^{n/2}$.

If n is odd.
If n is odd then the $\frac{n-1}{2}$ leftmost (rightmost) positions will take on $2^{(n-1)/2}$ values. The rightmost (leftmost) $\frac{n-1}{2}$ positions will take on the **same** values as the leftmost (rightmost) positions. Position

$\frac{n+1}{2}$ will take on 2 values.

If n = 3: (2)(2)(1) = 4 4 possible values.
If n = 5: (2)(2)(2)(1)(1) = 8 8 possible values.
If n = 7: (2)(2)(2)(2)(1)(1)(1) = 16 16 possible values.

If n odd: $2^{(n+1)/2}$.
∴,

Total number of Palindromes: $\begin{cases} 2^{n/2} & if\ n\ even \\ 2^{(n+1)/2} & if\ n\ odd \end{cases}$

OR

Total number of Palindromes: $2^{\lfloor (n+1)/2 \rfloor}$.

13. Use Venn Diagram with sets J, M, H.

JMH = 0.34
JM = 0.52
JH = 0.46
MH = 0.44
J$(\overline{M \cup H})$ = 0.1
$M(\overline{J \cup H})$ = 0.08
$H(\overline{M \cup J})$ = 0.06
$\overline{J \cup M \cup H}$ = 0.02
JM$\bar{H}$ = 0.18
JH$\bar{M}$ = 0.12
MH$\bar{J}$ = 0.1

A. P(M|J) = $\frac{P(MJ)}{P(J)} = \frac{0.52}{0.74} = \frac{26}{37}$.

B. P(H|JM) = $\frac{P(HJM)}{P(JM)} = \frac{0.34}{0.52} = \frac{17}{26}$.

C. P($\bar{J} \mid (M \cup H)\big) = \frac{P(\bar{J}(M \cup H))}{P(M \cup H)} = \frac{0.24}{0.88} = \frac{3}{11}$.

D. P(HJ | $\bar{M}$) = $\frac{P(HJ\bar{M})}{P(\bar{M})} = \frac{0.12}{0.3} = \frac{2}{5}$.

14. A. P(0 defectives) = $\frac{{}_{19}C_7}{{}_{24}C_7}$ = 0.145.

B. P(2 defectives) = $\frac{\left({}_{5}C_{2}\right)\left({}_{19}C_{5}\right)}{{}_{24}C_{7}}$ = 0.336.

C. P(at least 1 defective) = 1 – P(0 defectives)

P(at least 1 defective) = 1 – 0.145 = 0.855.

15. $\left({}_{10}C_{2}\right)\left({}_{10}C_{2}\right) = \binom{10}{2}\binom{10}{2} = 2025.$

OR

(N || lines) ⊥ (M || lines) = $\binom{N}{2}\binom{M}{2}$.

16. Let F: event of passing test on first try.

S: event of passing test on second try.

T: event of passing test on third try.

P(F) = 0.4

P(S|$\bar{F}$) = 0.6

P(T|$\bar{F}$ $\bar{S}$) = 0.2.

A. P(F ∪ $\bar{F}S$) = P(F) + P($\bar{F}$S) – P(F ∩ $\bar{F}$S)

= P(F) + P(S|$\bar{F}$) P($\bar{F}$) – P(F ∩ $\bar{F}$S)

= 0.4 + (0.6)(0.6) – 0

P(F ∪ $\bar{F}S$) = 0.76.

B. P($\bar{T}$ $\bar{S}$ $\bar{F}$) = P($\bar{T}$| $\bar{S}$ $\bar{F}$) P($\bar{S}$ $\bar{F}$)

$= P(\bar{T}|\ \bar{S}\ \bar{F})\ P(\bar{S}\ |\ \bar{F})\ P(\bar{F})$

$= (0.8)(0.4)(0.6)$

$P(\bar{T}\ \bar{S}\ \bar{F}) = 0.192.$

C. $P(T\ \bar{S}\ \bar{F}) = P(T|\ \bar{S}\ \bar{F})\ P(\bar{S}\ \bar{F})$

$= P(T|\ \bar{S}\ \bar{F})\ P(\bar{S}\ |\ \bar{F})\ P(\bar{F})$

$= (0.2)(0.4)(0.6)$

$P(\bar{T}\ \bar{S}\ \bar{F}) = 0.048.$

17. Let A: the set of positive integers which are divisible by 24.
B: the set of positive integers which are divisible by 36.

$P(A \cup B)$?

$P(A \cup B) = P(A) + P(B) - P(A \cap B).$

$|A| = \left\lfloor \frac{1000}{24} \right\rfloor = 41.$

$|B| = \left\lfloor \frac{1000}{36} \right\rfloor = 27.$

$|A \cap B| = \left\lfloor \frac{1000}{L.C.M.(24,36)} \right\rfloor = 13.$

∴,

$$P(A \cup B) = \frac{41}{1000} + \frac{27}{1000} - \frac{13}{1000} = \frac{55}{1000} = \frac{11}{200}.$$

NOTA BENE: L.C.M.(24, 36) = 72.

Solution to problem 4.d.

The probability that at least one bin is empty is equal to the sum of the probability that exactly one bin is empty and the probability that exactly two bins are empty.

The probability that exactly one bin is empty is: $\frac{(3)(2^5 - 2)}{3^5} = \frac{90}{243}$.

The probability that exactly two bins are empty is: $\frac{3}{3^5} = \frac{3}{243}$.

$\therefore$,

The probability that at least one bin is empty is: $\frac{90}{243} + \frac{3}{243} = \frac{93}{243}$.

Hence,

The probability that no bin is empty is: $1 - \frac{93}{243} = \frac{150}{243} \approx 0.617.$