Department of Computer Science & Software Engineering

Comp232 Mathematics for Computer Science

Fall 2018

Assignment 4.

1. Establish the following properties by induction or strong induction.

   (a) $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ for all $n \geq 1$.

   Solution.

   Let $P(n)$ be the proposition such that: $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ for all $n \geq 1$.

   Basis Step: $P(1)$ is true because LHS($P(1)$) = 1 and RHS($P(1)$) = $\frac{1 \times 2^2}{4} = 1$

   Inductive Step: To carry out the inductive step using this assumption, we must show that when we assume that $P(n)$ is true, then $P(n+1)$ is also true. That is, we must show that assuming the inductive hypothesis that $P(n+1)$ is also true. That is, we must show that

   $$\sum_{k=1}^{n+1} k^3 = \frac{(n+1)^2(n+2)^2}{4}$$

   assuming the inductive hypothesis for $P(n)$. Under the assumption of $P(n)$, we see that:

   $$\begin{aligned}
   \sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3 \\
   &= \frac{n^2(n+1)^2}{4} + (n+1)^3 && \text{using the inductive hypothesis for } P(n) \\
   &= \frac{(n+1)^2}{4}(n^2 + 4(n+1)) && \text{by factoring with } \frac{(n+1)^2}{4} \\
   &= \frac{(n+1)^2}{4} \times (n+2)^2 && \text{by factoring the last term} \\
   &= \frac{(n+1)^2(n+2)^2}{4}
   \end{aligned}$$

   Because we have completed the basis step and the inductive step, by mathematical induction, we know that $P(n)$ is true for all non negative integers $n \geq 1$.

   (b) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$ for all $n \geq 0$

   Solution.

   Let $P(n)$ be the proposition such that: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$ for all $n \geq 0$

   Basis Step: $P(0)$ is true because LHS($P(0)$) = 1 and RHS($P(0)$) = $1 - 0 = 1$, therefore LHS($P(0)$) > RHS($P(0)$).

Inductive Step: To carry out the inductive step using this assumption, we must show that when we assume that $P(n)$ is true, then $P(n+1)$ is also true. That is, we must show that assuming the inductive hypothesis that $P(n+1)$ is also true. That is, we must show that

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\cdots+\frac{1}{2^n}+\frac{1}{2^n+1}+\frac{1}{2^n+2}+\cdots+\frac{1}{2^{n+1}}\geq 1+\frac{n+1}{2}$$

assuming the inductive hypothesis for $P(n)$.

Under the assumption of $P(n)$, we see that:

$$\begin{aligned}
& 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\cdots+\frac{1}{2^n}+\frac{1}{2^n+1}+\frac{1}{2^n+2}+\cdots+\frac{1}{2^{n+1}} & \\
\geq\ & 1+\frac{n}{2}+\frac{1}{2^n+1}+\frac{1}{2^n+2}+\cdots+\frac{1}{2^{n+1}} & \text{using the inductive hypothesis}\\
\geq\ & 1+\frac{n}{2}+\frac{2^n}{2^{n+1}} & \text{lower bounding each fraction by } \frac{1}{2^{n+1}}\\
=\ & 1+\frac{n}{2}+\frac{1}{2}=1+\frac{n+1}{2} &
\end{aligned}$$

Because we have completed the basis step and the inductive step, by mathematical induction, we know that $P(n)$ is true for all non negative integers $n \geq 1$.

(c) Every positive integer n can be represented as a sum of distinct powers of 2, i.e., in the form $n = 2^{i_1} + \cdots + 2^{i_h}$ with integers $0 \leq i_1 < \cdots < i_h$.

Solution.

$$P(n): \forall n, n \text{ can be written } n = 2^{i_1} + \cdots + 2^{i_h} \text{ with integers } 0 \leq i_1 < \cdots < i_h.$$

Strong Induction.

Basis Step: The statement is true for $n = 0$.

Inductive Step: To carry out the inductive step using this assumption, we must show that when we assume that we are given an $n \geq 1$ and that it is true for all m with $0 \leq m < n$. That is, we must show that when we assume that $P(m)$ is true for $0 \leq m < n$, then $P(n)$ is also true.

When $n = 2m$ then $m = \frac{n}{2}$ and therefore, using the inductive hypothesis for $\frac{n}{2}$, $m = \frac{n}{2} = \sum_k 2^{p_k}$ with finitely many p_k, all of them different. It follows that $n = \sum_k 2^{i_k+1}$ with all $i_k + 1$ different.

When $n = 2m+1$, then $m = \frac{n-1}{2}$, and therefore, using the inductive hypothesis for $\frac{n-1}{2}$, we get with a similar reasoning as for the previous case that $n = 2^0 + \sum_k 2^{i_k+1}$ with all $i_k + 1$ different and different from 0.

(d) Let D_n denote the number of ways to cover the squares of a 2-by-n board using plain dominos. Then it is easy to see that $D_1 = 1, D_2 = 2$, and $D_3 = 3$. Compute a few more values of D_n and guess an expression for the value of D_n and use induction to prove you are right.

Solution.

$$D_4 = 5, D_5 = 8 \qquad \rightsquigarrow \text{(guess)} \qquad D_n = D_{n-1} + D_{n-2}$$

Proof by strong induction

Basis Step: The statement is true for $n = 3$: $D_3 = D_2 + D_1$

Inductive Step: Assume that we are given an $n \geq 3$ and that it is true for all m with $0 \leq m < n$. Let us consider an $2 \times n$ board.

The upper-right square of the board can be covered by a domino that is either laid horizontally or vertically.

- If covered by a vertically-laid domino, this leaves a $2 \times (n-1)$ grid that can be covered in D_{n-1} ways.
- If covered by a horizontally-laid domino, the domino below it must also lie horizontally. This leaves a $2 \times (n-2)$ grid that can be covered in D_{n-2} ways.

Because those are all the cases, we have proven that $D_n = D_{n-1} + D_{n-2}$.

2. Determine whether or not each of the following relations is a partial order and state whether or not each partial order is a total order.

   (a) $(\mathbb{N} \times \mathbb{N}, \preceq)$, where $(a, b) \preceq (c, d)$ if and only if $a \leq c$.
   Solution. This is not a partial order because the relation is not antisymmetric; for example, $(1, 4) \preceq (1, 8)$ because $1 \leq 1$ and similarly, $(1, 8) \preceq (1, 4)$, but $(1, 4) \neq (1, 8)$.

   (b) $(\mathbb{N} \times \mathbb{N}, \preceq)$, where $(a, b) \preceq (c, d)$ if and only if $a \leq c$ and $b \geq d$.
   Solution.
   This is a partial order.
   Reflexive: For any $(a, b) \in \mathbb{N} \times \mathbb{N}, (a, b) \preceq (a, b)$ because $a \leq a$ and $b \geq b$.
   Antisymmetric: If $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}, (a, b) \preceq (c, d)$ and $(c, d) \preceq (a, b)$, then $a \leq c$,

$b \geq d, c \leq a$ and $d \geq b$. So $a = c, b = d$ and hence, $(a, b) = (c, d)$.
Transitive: If $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{N}, (a, b) \preceq (c, d)$ and $(c, d) \preceq (e, f)$, then $a \leq c$, $b \geq d, c \leq e$ and $d \geq f$. So $a \leq e$ (because $a \leq c \leq e$) and $b \geq f$ (because $b \geq d \geq f$) and, therefore, $(a, b) \preceq (e, f)$.

This is not a total order; for example, (1, 4) and (2, 5) are incomparable.

3. Which of the following relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

   (a) $\{(a, b) | a$ and b are the same age$\}$

   (b) $\{(a, b) | a$ and b have the same parents$\}$

   (c) $\{(a, b) | a$ and b share a common parent$\}$

   (d) $\{(a, b) | a$ and b have met$\}$

   (e) $\{(a, b) | a$ and b speak a common language$\}$

   Solution.

   (a) is an equivalence relation,
   (b) is an equivalence relation,
   (c) is not an equivalence relation, not transitive,
   (d) is not an equivalence relation, not transitive,
   (e) is not an equivalence relation, not transitive.

4. Consider the following relation $\simeq$ defined on the set $\mathbb{N} \times \mathbb{Z}^+$.

$$(m_1, n_1) \simeq (m_2, n_2) \text{ iff } m_1 n_2 = m_2 n_1.$$

   (a) Prove that it is an equivalence and find equivalence classes.

   Solution.
   Reflexive. $(m_1, n_1) \simeq (m_1, n_1)$ iff $m_1 n_1 = m_1 n_1$.
   Symmetric. We have:

$$(m_1, n_1) \simeq (m_2, n_2) \text{ iff } m_1 n_2 = m_2 n_1.$$
$$(m_2, n_2) \simeq (m_1, n_1) \text{ iff } m_2 n_1 = m_1 n_2.$$

   therefore $(m_1, n_1) \simeq (m_2, n_2)$ whenever $(m_2, n_2) \simeq (m_1, n_1)$
   Transitive.

$$(m_1, n_1) \simeq (m_2, n_2) \text{ iff } m_1 n_2 = m_2 n_1.$$
$$(m_2, n_2) \simeq (m_3, n_3) \text{ iff } m_2 n_3 = m_3 n_2.$$

therefore

$$\begin{aligned}
m_2(m_1n_3) &= (m_2m_1)n_3 && \text{commutativity} \\
&= (m_1m_2)n_3 && \text{commutativity of multiplication} \\
&= m_1(m_2n_3) && \text{associativity of multiplication} \\
&= m_1(n_2m_3) && \text{as } (m_2,n_2) \simeq (m_3,n_3) \\
&= (m_1n_2)m_3 && \text{associativity of multiplication} \\
&= (n_1m_2)m_3 && \text{as } (m_1,n_1) \simeq (m_2,n_2) \\
&= (m_2n_1)m_3 && \text{commutativity of multiplication} \\
&= m_2(n_1m_3) && \text{associativity of multiplication}
\end{aligned}$$

It follows that $n_1m_3 = m_1n_3$ as $m_2 \neq 0$, i.e., $(m_1,n_1) \simeq (m_3,n_3)$

(b) Provide a concise characterization of the equivalence classes in terms of rational numbers.

Solution.
There is one equivalence class for each distinct rational number. Each equivalence class consists of all ordered pairs (a,b) that, if written as fractions $\frac{a}{b}$, would equal each other. Equivalence class of rational $r = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} | y \neq 0 \text{ and } \frac{x}{y} = r\}$.

5. Consider the following relation $\prec$ over reals: $x \prec y$ iff $(x-y) \in \mathbb{Z}$. Prove that it is an equivalence and characterize its equivalence classes

Solution.

Reflexive. To see that $\prec$ is reflexive, let $x \in \mathbb{R}$. Then $x - x = 0$ and $0 \in \mathbb{Z}$, so $x \prec x$.

Symmetric. To see that $\prec$ is symmetric, let $a, b \in \mathbb{R}$. Suppose $a \prec b$. Then $a - b \in \mathbb{Z}$ say $a - b = m$, where $m \in \mathbb{Z}$. Then $b - a = -(a-b) = -m$ and $-m \in \mathbb{Z}$. Thus, $b \prec a$.

Transitive. To see that $\prec$ is transitive, let $a, b, c \in \mathbb{R}$. Suppose that $a \prec b$ and $b \prec c$. Thus, $a - b \in \mathbb{Z}$, and $b - c \in \mathbb{Z}$. Suppose $a - b = m$ and $b - c = n$, where $m, n \in \mathbb{Z}$. Then $a - c = (a-b) + (b-c) = m + n$. Now $m + n \in \mathbb{Z}$; that is, $a - c \in \mathbb{Z}$. Therefore $a \prec c$.

Equivalence classes. Let the above relation be called R.

$$\begin{aligned}
[a]_R &= \{b \in \mathbb{R} \mid (a,b) | a - b = kn \text{ for some integer } k\} \\
&= \{b \in \mathbb{R} \mid a \text{ and } b \text{ have the same decimal part}\}.
\end{aligned}$$

6. A set S of jobs can be ordered by writing $x \leq y$ to mean that either $x = y$ or x must be done before y, for all x and y in S. Given the Hasse diagram represented in Figure 1 for this relation for a particular set S of jobs, show the following:

   (a) minimal, least, maximal, and greatest elements;

   (b) a topological sort.

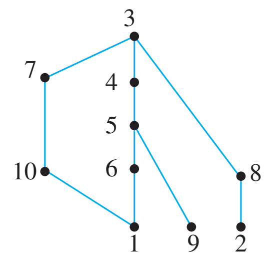


Figure 1: Hasse Diagram

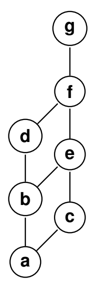
Solution.

Reminder. There is usually not a unique way to define a topological sort.

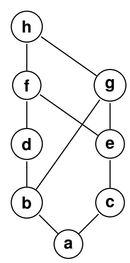
Minimal = 1, 2, 9.
Least does not exist.
Maximal = Greatest = 3.
A topological sort requires to iteratively choose one of the minimal elements as least, e.g.,

$$1 \leq 9 \leq 2 \leq 10 \leq 6 \leq 8 \leq 5 \leq 7 \leq 4 \leq 3.$$

7. Determine whether the posets with these Hasse diagrams are lattices.



Poset 1



Poset 2

Solution.
Poset 1: Yes. Every two elements will have a least upper bound and greatest lower bound
Poset 2: No. If we take the elements b and c, then we will have f, g, and h as the upper bound, but none of them will be the least upper bound

8. Determine whether the following posets are lattices:

(a) $(1, 3, 6, 9, 12, |)$

(b) $(1, 5, 25, 125, |)$

(c) $(\mathbb{Z}, \geq)$

(d) $(P(S), \supseteq)$, where $P(S)$ is the power set of a set S.

Solution. In each case, we need to decide whether every pair of elements has a least upper bound and a greatest lower bound.

(a) This is not a lattice, since the element 6 and 9 have no upper bound.

(b) This is a lattice since it is a linear order.

(c) This is a lattice since it is a linear order.

(d) This is a lattice since, for any pair of elements A and B of $P(S)$, the GLB (Greatest Lower Bound) is $A \cup B$ and the LUB (Least Upper Bound) is $A \cap B$

9. Let R be a relation on $\mathbb{N}$ defined by $(x, y) \in R$ iff there is a prime p such that $y = px$. Describe in words the reflexive, symmetric and transitive closures of R, denoted by r, s and t, respectively.

Solution.

Remember:
0 and 1 are **not** prime numbers.
$\mathbb{N} = \{0, 1, 2, \dots\}$ set of natural numbers

$R = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid \exists$ prime p such that $y = px\}$.

$r(R)$ = reflexive closure of R
$= \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid$ there is a prime p such that $y = px\} \cup \{(x, x)\}$
$= \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid$ either there is a prime p or $p = 1$ such that $y = px\}$

$s(R)$ = symmetric closure of R
$= \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid$ there is a prime p such that $y = px\} \cup \{(y, x) \mid$ there is a prime p such that $y = px\}$
$= \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid$ either there is a prime p such that $y = px$ or $x = py\}$

$t(R)$ = transitive closure of R
$= \{(x, z) \in \mathbb{N} \times \mathbb{N} \mid \exists y$ such that $(x, y) \in R$ and $(y, z) \in R\}$
$= \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x|y$ for $x < y\} \cup \{(0, 0)\}$.
Note that $(x, y) \in R$ can be interpreted as: either $x = y = 0$ (observe that if $x = 0$, $y = 0$ as a prime cannot be 0), or x divides y.

(a) Which of the following are true:

$r(s(R)) = s(r(R))$
$r(t(R)) = t(r(R))$
$s(t(R)) = t(s(R))$

You need to justify your answer.
Solution.

- $r(s(R)) = s(r(R))$: True
  Indeed, $r(s(R)) = s(r(R)) = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid$ either there is a prime p such that $y = px$ or $x = py\} \cup \{(x, x) \mid x \in \mathbb{N}\}$
- $r(t(R)) = t(r(R))$: True
  Indeed, $r(t(R)) = t(r(R)) = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x|y$ for $x < y\} \cup \{(x, x) \mid x \in \mathbb{N}\}$
- $s(t(R)) = t(s(R))$: False
  Note that $(x, x) \notin R$. However, $(x, x) \in t(s(R))$ but $(x, x) \notin s(t(R))$
  $(x, x) \in t(s(R))$: If $(x, y) \in R$, then $(y, x) \in s(R)$, and consequently $(x, x) \in t(s(R))$.
  On the other hand, $(x, x) \notin t(R)$, and therefore it cannot belong to $s(t(R))$.

(b) Which of them hold for all relations on $\mathbb{N}$?

Solution. None

(c) Using the reflexive, symmetric, and transitive closures, express the smallest equivalence relation containing an arbitrary relation.

Solution.

If we have a relation R that does not satisfy a property P (such as reflexivity or symmetry), we can add edges until it does. This is called the P closure of R. Consequently, a closure gives the smallest possible relation with property P.

In order to get the smallest equivalence relation, we need to take all three closures (reflexive, symmetric, transitive).

For an arbitrary relation S: $r(s(t(S)))$

For relation R: it is defined by relation T such that

$$T = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x|y \text{ or } y|x\} \cup \{(0,0)\}$$

(d) What is the smallest partial order containing R?

Solution. Since the relation is already asymmetric, we only need to consider the reflexive and transitive closure in order to get the smallest partial order containing R.

$$T = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x|y\} \cup \{(0,0)\}.$$