# Department of Computer Science & Software Engineering

# Comp232 Mathematics for Computer Science

## Fall 2018

## Assignment 4.

1. Establish the following properties by induction or strong induction.

   (a) $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ for all $n \geq 1$.

   (b) $1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{2^n} \geq 1 + \frac{n}{2}$ for all $n \geq 0$

   (c) Every positive integer $n$ can be represented as a sum of distinct powers of 2, i.e., in the form $n = 2^{i_1} + \cdots + 2^{i_h}$ with integers $0 \leq i_1 < \cdots < i_h$.

   (d) Let $D_n$ denote the number of ways to cover the squares of a 2-by-$n$ board using plain dominos. Then it is easy to see that $D_1 = 1, D_2 = 2$, and $D_3 = 3$. Compute a few more values of $D_n$ and guess an expression for the value of $D_n$ and use induction to prove you are right.

2. Determine whether or not each of the following relations is a partial order and state whether or not each partial order is a total order.

   (a) $(\mathbb{N} \times \mathbb{N}, \preceq)$, where $(a,b) \preceq (c,d)$ if and only if $a \leq c$.

   (b) $(\mathbb{N} \times \mathbb{N}, \preceq)$, where $(a,b) \preceq (c,d)$ if and only if $a \leq c$ and $b \geq d$.

3. Which of the following relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

   (a) $\{(a,b) | a$ and $b$ are the same age$\}$

   (b) $\{(a,b) | a$ and $b$ have the same parents$\}$

   (c) $\{(a,b) | a$ and $b$ share a common parent$\}$

   (d) $\{(a,b) | a$ and $b$ have met$\}$

   (e) $\{(a,b) | a$ and $b$ speak a common language$\}$

4. Consider the following relation $\simeq$ defined on the set $\mathbb{N} \times \mathbb{Z}^+$.

   $$(m_1, n_1) \simeq (m_2, n_2) \text{ iff } m_1 n_2 = m_2 n_1.$$

   (a) Prove that it is an equivalence and find equivalence classes.

   (b) Provide a concise characterization of the equivalence classes in terms of rational numbers

5. Consider the following relation $\prec$ over reals: $x \prec y$ iff $(x - y) \in \mathbb{Z}$. Prove that it is an equivalence and characterize its equivalence classes

6. A set $S$ of jobs can be ordered by writing $x \leq y$ to mean that either $x = y$ or $x$ must be done before $y$, for all $x$ and $y$ in $S$. Given the Hasse diagram represented in Figure 1 for this relation for a particular set $S$ of jobs, show the following:

   (a) minimal, least, maximal, and greatest elements;

   (b) a topological sort.

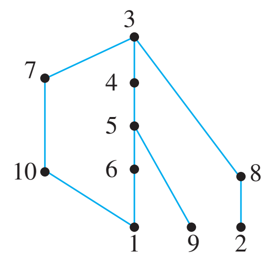
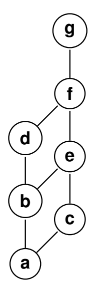
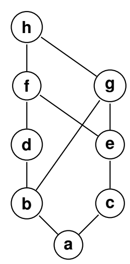



Figure 1: Hasse Diagram

7. Determine whether the posets with these Hasse diagrams are lattices.

Poset 1 


Poset 2 


8. Determine whether the following posets are lattices:

   (a) $(1, 3, 6, 9, 12, |)$

   (b) $(1, 5, 25, 125, |)$

   (c) $(\mathbb{Z}, \geq)$

   (d) $(P(S), \supseteq)$, where $P(S)$ is the power set of a set $S$.

9. Let $R$ be a relation on $\mathbb{N}$ defined by $(x, y) \in R$ iff there is a prime $p$ such that $y = px$. Describe in words the reflexive, symmetric and transitive closures of $R$, denoted by $r, s$ and $t$, respectively.

   (a) Which of the following are true:

   $$r(s(R)) = s(r(R))$$
   $$r(t(R)) = t(r(R))$$
   $$s(t(R)) = t(s(R))$$

   You need to justify your answer.

   (b) Which of them hold for all relations on $\mathbb{N}$?

   (c) Using the reflexive, symmetric, and transitive closures, express the smallest equivalence relation containing an arbitrary relation.

   (d) What is the smallest partial order containing $R$?