Concordia University

Department of Computer Science & Software Engineering

COMP 232/2 Mathematics for Computer Science 2018-19

Assignment 2

Due date: 12.10.2018

In each of the problems below it is especially important that your proof (or counter example) is correct, clear, complete, concise, and carefully presented, using proper mathematical notation. Justifications have to be given in the format of valid arguments. Points will be deducted if your presentation does not satisfy these requirements.

1. **Answer:**
   Knaves: Alice, Bob, Diane. Knights: Earl, Cecil, Fiona

   Proof by cases:

   Case 1: Earl is a knave.

   Then Alice is also a knave.
   Thus Fiona is a knave.
   Then Bob is a knave (from what Fiona said).
   This leads to a contradiction, since Bob tells the truth under these assumptions!

   Thus Earl cannot be a knave.

   Case 2: Earl is a knight.

   Then Alice is a knave.
   Fiona tells the truth and is a knight.
   Diane tells a falsehood and is a knave.
   Cecil tells the truth and is a knight.
   Bob tells a falsehood and is a knave.

   These are consistent assignments.

   Note that we didn't have to conduct a proof by cases, because of the excluded middle in logic, we could have simply provided the correct assignments. This was meant to show that orderly proof techniques can help find the solution, not only present it.

2. **Answer:**

S1: Universal instantiation:
$\forall x \ (P(x) \rightarrow Q(x))$ and $c \in D$
$P(c) \rightarrow Q(c)$

S2: Universal instantiation:
$\forall x \ (Q(x) \rightarrow R(x))$
$Q(c) \rightarrow R(c)$

S3: Hypothetical syllogism:
$P(c) \rightarrow Q(c)$
$Q(c) \rightarrow R(c)$

$P(c) \rightarrow R(c)$

S4: Universal generalization:
$P(c) \rightarrow R(c)$ for an arbitrary c of domain D

$\forall x \ (P(x) \rightarrow R(x))$ as desired

3. (a) **Answer:**

$$\begin{aligned}
&[(p \vee q) \wedge (p \rightarrow s) \wedge (q \rightarrow t)] \longrightarrow (s \vee t) && \\
&\equiv \neg[(p \vee q) \wedge (\neg p \vee s) \wedge (\neg q \vee t)] \vee (s \vee t) && \text{def. conditional} \\
&\equiv [\neg(p \vee q) \vee \neg(\neg p \vee s) \vee \neg(\neg q \vee t)] \vee (s \vee t) && \text{De Morgan} \\
&\equiv [(\neg p \wedge \neg q) \vee (p \wedge \neg s) \vee (q \wedge \neg t)] \vee (s \vee t) && \text{De Morgan} \\
&\equiv (\neg p \wedge \neg q) \vee (p \wedge \neg s) \vee (q \wedge \neg t) \vee s \vee t && \text{extended notion of associativity} \\
&\equiv (\neg p \wedge \neg q) \vee ((p \wedge \neg s) \vee s) \vee ((q \wedge \neg t) \vee t) && \text{extended notion of associativity and commutativity} \\
&\equiv (\neg p \wedge \neg q) \vee ((p \vee s) \wedge (\neg s \vee s)) \vee ((q \vee t) \wedge (\neg t \vee t)) && \text{distributive law} \\
&\equiv (\neg p \wedge \neg q) \vee ((p \vee s) \wedge \text{TRUE}) \vee ((q \vee t) \wedge \text{TRUE}) && \text{def. } \vee \\
&\equiv (\neg p \wedge \neg q) \vee ((p \vee s)) \vee ((q \vee t)) && \text{identity} \\
&\equiv (\neg p \wedge \neg q) \vee p \vee s \vee q \vee t && \text{associativity} \\
&\equiv (\neg p \wedge \neg q) \vee (p \vee q) \vee s \vee t && \text{associativity, commutativity} \\
&\equiv \neg(p \vee q) \vee (p \vee q) \vee s \vee t && \text{De Morgan} \\
&\equiv \text{TRUE} \vee s \vee t && \text{def. } \vee \\
&\equiv \text{TRUE} && \text{domination}
\end{aligned}$$

(b) **Answer:** Direct proof

Assume that $((p \vee q) \wedge (p \rightarrow s) \wedge (q \rightarrow t)) \equiv T$

Then
$(p \vee q) \equiv T$
$(p \rightarrow s) \equiv T$
$(q \rightarrow t) \equiv T$.

Case 1: $p \equiv T$

Then
from p

and $p \to s$

follows s

Case 2: $q \equiv T$

Then
from q
and $q \to t$)

follows t

Since $p \vee q$, $(s \vee t) \equiv T$, as desired.

(c) **Answer:** Proof by contrapositive

Assume: $\neg(s \vee t) \equiv \neg s \wedge \neg t$

Case 1: Let $(\neg p \wedge \neg q) \equiv T$
Case 2: Let $p \equiv T$. Then by addition: $(p \wedge \neg s) \equiv T$
Case 3: Let $q \equiv T$ Then by addition: $(q \wedge \neg t) \equiv T$

Since one of Cases 1-3 has to be true,
it follows that
$(\neg p \wedge \neg q) \vee (p \wedge \neg s) \vee (q \wedge \neg t)$
$\equiv \neg\left(\neg(\neg p \wedge \neg q) \wedge \neg(p \wedge \neg s) \wedge \neg(q \wedge \neg t)\right)$ Double negation
$\equiv \neg\left((p \vee q) \wedge (\neg p \vee s) \wedge (\neg q \vee t)\right)$
$\equiv \neg\left((p \vee q) \wedge (p \to s) \wedge (q \to t)\right)$

We have thus shown that assuming $\neg(s \vee t)$ logically leads to $\neg\left((p \vee q) \wedge (p \to s) \wedge (q \to t)\right)$, which concludes the proof by contrapositive.

4. (a) **Answer:** Proof by contrapositive

Assume n is even , then $n = 2k$ for some integer k

Then $n^3 + 3n + 1 = (2k)^3 + 3(2k) + 1 = 8k^3 + 6k + 1 = 2(4k^3 + 3k) + 1 = 2j + 1$ for some integer $j = 4k^3 + 3k$

Thus if we assume that n is even, it follows logically that $n^3 + 3n + 1$ is odd, which concludes the proof by contrapositive.

(b) **Answer:** Proof by contrapositive

Assume x is rational, then there are integers a, b with no common factors, $b \neq 0$, such that $x = \frac{a}{b}$.
Let $x^3 + 3x + 3 = k$

Then:

$$x^3 + 3x + 3 = \left(\frac{a}{b}\right)^3 + 3\frac{a}{b} + 3 = \frac{a^3}{b^3} + \frac{3a}{b} + \frac{3}{1} = \frac{a^3}{b^3} + \frac{3ab^2}{bb^2} + \frac{3b^3}{b^3} = \frac{a^3}{b^3} + \frac{3ab^2}{b^3} + \frac{3b^3}{b^3} = \frac{a^3+3ab^2+3b^3}{b^3}$$

Since $a^3 + 3ab^2 + 3b^3$ and b^3 must each be integers, the fraction represents a rational, as desired.

5. **Answer:** Proof by contradiction:

   Assume that every triple of adjacent integers placed randomly around a circle has a sum less than or equal to 32.

   Now, excluding 1 (which must be placed somewhere), there remain exactly three groups of triples of adjacent integers.

   The total sum is then (no greater than) 1 + 32 + 32 + 32 = 97.

   However, the sum of 1, 3, 5, 7, 9, 11, 13, 15, 17, and 19, should be 100.

   This is a contradiction.

6. **Answer** Proof by cases

   Case 1: let $n = 2k$ for some integer k (n is even).

   Then $n(n^2 - 1)(n + 2) = n^4 + 2n^3 - n^2 - 2n = 16k^4 + 8k^3 - 4k^2 - 4k = 4(4k^4 + 2k^3 - k^2 - k$

   Case 2: let $n = 2k + 1$ for some integer k (n is odd).

   Then $n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k^2 + 4k = 4(k^2 + k)$,

   Thus $n(n^2 - 1)(n + 2) = n\ 4(k^2 + k)\ (n + 2) = 4(n)(k^2 + k)(n + 2) = 4j$ for some integer j.