$\color{red}{(}a+b\color{red}{)}$

Department of Computer Science & Software Engineering

Comp232 Mathematics for Computer Science

Fall 2018

Assignment 1. Solutions

1. For each of the following statements use a truth table to determine whether it is a tautology, a contradiction, or a contingency.

   (a) $\Big(\underbrace{(p \vee r)}_{\mathbf{a}} \wedge \underbrace{(q \vee r)}_{\mathbf{b}}\Big) \leftrightarrow \underbrace{\Big((p \wedge q) \vee r\Big)}_{\mathbf{c}}$

   Solution: Tautology.

p	q	r	$\overbrace{p \vee r}^{\mathbf{a}}$	$\overbrace{q \vee r}^{\mathbf{b}}$	$\mathbf{a} \wedge \mathbf{b}$	$p \wedge q$	$\overbrace{(p \wedge q) \vee r}^{\mathbf{c}}$	$(\mathbf{a} \wedge \mathbf{b}) \leftrightarrow \mathbf{c}$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	T	T	T	F	T	T
F	T	T	T	T	T	F	T	T
T	F	F	T	F	F	F	F	T
F	T	F	F	T	F	F	F	T
F	F	T	T	T	T	F	T	T
F	F	F	F	F	F	F	F	T

   (b) $(p \oplus q) \wedge (p \oplus \neg q)$

   Solution: Contradiction.

p	q	$\neg q$	$\overbrace{p \oplus q}^{\mathbf{a}}$	$\overbrace{p \oplus \neg q}^{\mathbf{b}}$	$\mathbf{a} \wedge \mathbf{b}$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	F

(c) $\big(p \to (q \to r)\big) \leftrightarrow \big(p \to (q \wedge r)\big)$

Solution: Contingency.

p	q	r	$\overbrace{q \to r}^{\mathbf{a}}$	$\overbrace{q \wedge r}^{\mathbf{b}}$	$\overbrace{p \to \mathbf{a}}^{\mathbf{c}}$	$\overbrace{p \to \mathbf{b}}^{\mathbf{d}}$	$\mathbf{c} \leftrightarrow \mathbf{d}$
T	T	T	T	T	T	T	T
T	T	F	F	F	F	F	T
T	F	T	T	F	T	F	F
F	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	F	F	F	T	T	T
F	F	T	T	F	T	T	T
F	F	F	T	F	T	T	T

(d) $\underbrace{\big(p \wedge \underbrace{(\neg q \to \neg p)}_{\mathbf{a}}\big)}_{\mathbf{b}} \to q$

Solution: Tautology.

p	q	$\neg p$	$\neg q$	$\overbrace{\neg q \to \neg p}^{\mathbf{a}}$	$\overbrace{p \wedge \mathbf{a}}^{\mathbf{b}}$	$\mathbf{b} \to q$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

2. For each of the following logical equivalences state whether it is valid or invalid. If invalid then give a counterexample (*e.g.*, based on a truth table). If valid then give an algebraic proof using logical equivalences from Tables 6, 7, and 8 from Section 1.3 of textbook.

   (a) $(p \to r) \land (q \to r) \ \equiv\ (p \land q) \to r$

   Solution: Invalid.
   If $p = T$, $q = F$, and $r = F$ then the LHS is False, while the RHS is True.

   (b) $(p \to q) \lor (p \to r) \ \equiv\ (p \lor q) \to r$

   Solution: Invalid.
   If $p = T$, $q = T$, and $r = F$ then the LHS is True, while the RHS is False.

   (c) $\Big(\big((p \lor q) \land (p \to r) \land (q \to r)\big) \to r\Big) \ \equiv\ T$

   Solution: Valid.

$$
\begin{array}{rrl}
 & \Big(\big((p \lor q) \land (p \to r)\big) \land (q \to r)\Big) \to r & \text{Assumption} \\
\equiv & \neg\Big(\big((p \lor q) \land (p \to r)\big) \land (q \to r)\Big) \lor r & \text{Law for conditional} \\
\equiv & \Big(\neg\big((p \lor q) \land (p \to r)\big) \lor \neg(q \to r)\Big) \lor r & \text{De Morgan} \\
\equiv & \Big(\big(\neg(p \lor q) \lor \neg(p \to r)\big) \lor \neg(q \to r)\Big) \lor r & \text{De Morgan} \\
\equiv & \Big(\big(\neg(p \lor q) \lor \neg(\neg p \lor r)\big) \lor \neg(\neg q \lor r)\Big) \lor r & \text{Law for conditional, twice} \\
\equiv & \Big(\big((\neg p \land \neg q) \lor (\neg\neg p \land \neg r)\big) \lor (\neg\neg q \land \neg r)\Big) \lor r & \text{De Morgan, trice} \\
\equiv & \Big(\big((\neg p \land \neg q) \lor (p \land \neg r)\big) \lor (q \land \neg r)\Big) \lor r & \text{Double negation, twice} \\
\equiv & \Big((\neg p \land \neg q) \lor \big((p \land \neg r) \lor (q \land \neg r)\big)\Big) \lor r & \text{Associativity} \\
\equiv & \Big((\neg p \land \neg q) \lor \big((p \land q) \lor \neg r\big)\Big) \lor r & \text{Distributivity} \\
\equiv & \Big(\big((\neg p \land \neg q) \lor (p \land q)\big) \lor \neg r\Big) \lor r & \text{Associativity} \\
\equiv & \big((\neg p \land \neg q) \lor (p \land q)\big) \lor \Big(\neg r \lor r\Big) & \text{Associativity} \\
\equiv & \big((\neg p \land \neg q) \lor (p \land q)\big) \lor T & \text{Excluded middle} \\
\equiv & T & \text{Domination}
\end{array}
$$

(d) $\Big(\big((p \to q) \wedge (q \to r)\big) \to (p \to r)\Big) \equiv T$

Solution: Valid.
We shall instead prove $\neg\Big(\big((p \to q) \wedge (q \to r)\big) \to (p \to r)\Big) \equiv F$.
The claim then follows since $\neg A \equiv F$ if and only if $A \equiv T$.

$$
\begin{array}{rrl}
 & \neg\Big(\big((p \to q) \wedge (q \to r)\big) \to (p \to r)\Big) & \\
\equiv & \big((p \to q) \wedge (q \to r)\big) \wedge \neg(p \to r) & \text{Table 7, law 5} \\
\equiv & \big((p \to q) \wedge (q \to r)\big) \wedge (p \wedge \neg r) & \text{Table 7, law 5} \\
\equiv & \big((p \to q) \wedge (q \to r)\big) \wedge (\neg r \wedge p) & \text{commutativity} \\
\equiv & \Big(\big((p \to q) \wedge (q \to r)\big) \wedge \neg r\Big) \wedge p & \text{associativity} \\
\equiv & \Big((p \to q) \wedge \big((q \to r) \wedge \neg r\big)\Big) \wedge p & \text{associativity} \\
\equiv & \Big((p \to q) \wedge \big((\neg q \vee r) \wedge \neg r\big)\Big) \wedge p & \text{law for conditional} \\
\equiv & \Big((p \to q) \wedge \big((\neg q \wedge \neg r) \vee (r \wedge \neg r)\big)\Big) \wedge p & \text{distributivity} \\
\equiv & \Big((p \to q) \wedge \big((\neg q \wedge \neg r) \vee F\big)\Big) \wedge p & \text{excluded middle} \\
\equiv & \big((p \to q) \wedge (\neg q \wedge \neg r)\big) \wedge p & \text{identity} \\
\equiv & \big((\neg p \vee q) \wedge (\neg q \wedge \neg r)\big) \wedge p & \text{law for conditional} \\
\equiv & \big(((\neg p \vee q) \wedge \neg q) \wedge \neg r\big) \wedge p & \text{associativity} \\
\equiv & \big(((\neg p \wedge \neg q) \vee (q \wedge \neg q)) \wedge \neg r\big) \wedge p & \text{distributivity} \\
\equiv & \big(((\neg p \wedge \neg q) \vee F) \wedge \neg r\big) \wedge p & \text{excluded middle} \\
\equiv & \big((\neg p \wedge \neg q) \wedge \neg r\big) \wedge p & \text{identity} \\
\equiv & p \wedge \big((\neg p \wedge \neg q) \wedge \neg r\big) & \text{commutativity} \\
\equiv & p \wedge \big(\neg p \wedge (\neg q \wedge \neg r)\big) & \text{associativity} \\
\equiv & (p \wedge \neg p) \wedge (\neg q \wedge \neg r) & \text{associativity} \\
\equiv & F \wedge (\neg q \wedge \neg r) & \text{excluded middle} \\
\equiv & F & \text{domination}
\end{array}
$$

3. Which of the following conditions is *necessary* for the natural number n to be divisible by 6. The natural numbers are $\mathbb{N} = \{0, 1, 2, \ldots, \}$.

   (a) n is divisible by 3.

   (b) n is divisible by 9.

   (c) n is divisible by 12.

   (d) $n = 24$

   (e) n^2 is divisible by 3.

   (f) n is even and divisible by 3.

   Solution:

   Necessary means *If n is divisible by 6, then condition.* Conditions (a), (e), and (f) are necessary. We have

   (a) If n is divisible by 6, then n is divisible by 3.

   (e) If n is divisible by 6, then n^2 is divisible by 3.

   (f) If n is divisible by 6, then n is even and divisible by 3.

   Sufficient means *If condition, then n is divisible by 6.* Conditions (c), (d), and (f) are sufficient. We have

   (c) If n is divisible by 12, then n is divisible by 6.

   (d) If $n = 24$, then n is divisible by 6.

   (f) If n is even and divisible by 3, then n is divisible by 6.

   Note that (f) is both necessary and sufficient, meaning that

   n is divisible by 6 if and only if (iff) n is even and divisible by 3

4. A set of propositions is *consistent* if there is an assignment of truth values to each of the variables in the propositions that makes each proposition true. Is the following set of propositions consistent?

   (a) *If the file system is not locked, then new messages will be queued.*

   (b) *If the file system is not locked, then the system is functioning normally, and conversely.*

   (c) *If new messages are not queued, then they will be sent to the message buffer.*

   (d) *If the file system is not locked, then new messages will be sent to the message buffer.*

   (e) *New messages will not be sent to the message buffer.*

Solution:

Let us define the following propositions:

$FL =_{\texttt{def}}$ *The file system is locked.*

$NQ =_{\texttt{def}}$ *New messages will be queued.*

$FN =_{\texttt{def}}$ *The system is functioning normally.*

$NB =_{\texttt{def}}$ *New messages will be sent to the message buffer.*

We can now formalize propositions (a) – (e):

(a) $\neg FL \rightarrow NQ$

(b) $\neg FL \leftrightarrow FN$

(c) $\neg NQ \rightarrow NB$

(d) $\neg FL \rightarrow NB$

(e) $\neg NB$

The set (a) – (e) of propositions (the conjunction of the proposition in the set) is indeed satisfiable. A satisfying truth assignment is

$$FL = True,\ NQ = True,\ FN = False,\ NB = False$$

5. Suppose the domain of the propositional function $P(x, y)$ consists of pairs x and y, where x = 1, 2, or 3, and y = 1, 2, or 3. Write out the propositions below using disjunctions and conjunctions only.

   (a) $\exists x\, P(x, 3)$

   Solution: $P(1, 3) \vee P(2, 3) \vee P(3, 3)$

   (b) $\forall y\, \neg P(2, y)$

   Solution: $\neg P(2, 1) \wedge \neg P(2, 2) \wedge \neg P(2, 3)$

   (c) $\forall x\, \exists y\, P(x, y)$

   Solution:

$$
\begin{aligned}
& \forall x\, \exists y\, P(x, y) \\
\equiv \quad & \Big(\exists y\, P(1, y)\Big) \bigwedge \Big(\exists y\, P(2, y)\Big) \bigwedge \Big(\exists y\, P(3, y)\Big) \\
\equiv \quad & \Big(P(1, 1) \vee P(1, 2) \vee P(1, 3)\Big) \\
\bigwedge \quad & \Big(P(2, 1) \vee P(2, 2) \vee P(2, 3)\Big) \\
\bigwedge \quad & \Big(P(3, 1) \vee P(3, 2) \vee P(3, 3)\Big)
\end{aligned}
$$

   (d) $\exists x\, \forall y\, \neg P(x, y)$

   Solution:

$$
\begin{aligned}
& \exists x\, \forall y\, \neg P(x, y) \\
\equiv \quad & \Big(\forall y\, \neg P(1, y)\Big) \bigvee \Big(\forall y\, \neg P(2, y)\Big) \bigvee \Big(\forall y\, \neg P(3, y)\Big) \\
\equiv \quad & \Big(\neg P(1, 1) \wedge \neg P(1, 2) \wedge \neg P(1, 3)\Big) \\
\bigvee \quad & \Big(\neg P(2, 1) \wedge \neg P(2, 2) \wedge \neg P(2, 3)\Big) \\
\bigvee \quad & \Big(\neg P(3, 1) \wedge \neg P(3, 2) \wedge \neg P(3, 3)\Big)
\end{aligned}
$$

6. Let the domain for x and x' be the set of all students in this class and the domain for y be the set of all countries in the world. Let $P(x,y)$ denote *student x has visited country y* and $Q(x,y)$ denote *student x has a friend in country y*. Express each of the following using logical operations and quantifiers, and the propositional functions $P(x,y)$ and $Q(x,y)$.

   (a) *Carlos has visited Bulgaria.*

   Solution: $P(Carlos, Bulgaria)$

   (b) *Every student in this class has visited the United States.*

   Solution: $\forall x\, P(x, UnitedStates)$

   (c) *Every student in this class has visited some country in the world.*

   Solution: $\forall x\, \exists y\, P(x,y)$

   (d) *There is no country that every student in this class has visited.*

   Solution: $\forall y\, \exists x\, \neg P(x,y)$. Equivalently $\neg\Big(\exists y \forall x P(x,y)\Big)$

   (e) *There are two students in this class, who between them, have a friend in every country in the world.*

   Solution: $\exists x\, \exists x' \Big(x \neq x' \wedge \forall y[Q(x,y) \vee Q(x',y)]\Big)$

   (f) *Nobody in this class has visited a country in which they did not have a friend.*

   Solution: $\forall x\, \forall y \Big(P(x,y) \to Q(x,y)\Big)$

   Equivalent solution: $\neg\Big[\exists x\, \exists y \Big(P(x,y) \wedge \neg Q(x,y)\Big)\Big]$

7. For each part in the previous question, form the negation of the statement so that all negation symbols occur immediately in front of predicates. For example:

$$\neg\Big(\forall x\big(P(x) \wedge Q(x)\big)\Big) \equiv \exists x\Big(\neg\big((P(x) \wedge Q(x)\big)\Big) \equiv \exists x\Big(\big(\neg P(x)\big) \vee \big(\neg Q(x)\big)\Big)$$

(a) **Solution:** $\neg P(Carlos, Bulgaria)$

Carlos has not visited Bulgaria

(b) **Solution:** $\neg\Big(\forall x\ P(x, UnitedStates)\Big) \equiv \exists x\ \Big(\neg P(x, UnitedStates)\Big)$

There is a student in this class who has not visited the United States

(c) **Solution:** $\neg\ \Big(\forall x\ \Big[\exists y\ P(x,y)\Big]\Big) \equiv \exists x\ \neg\Big[\exists y\ P(x,y)\Big] \equiv \exists x\ \forall y\ \Big[\neg P(x,y)\Big]$

There is a student in this class who has not visited any country

(d) **Solution:**

$$\begin{aligned}
&\neg\ \Big(\forall y\ \Big[\exists x\ \neg P(x,y)\Big]\Big) && \equiv \\
&\exists y\ \neg\Big[\exists x \neg P(x,y)\Big] && \equiv \\
&\exists y\ \forall x\ \neg\Big[\neg P(x,y)\Big] && \equiv \\
&\exists y\ \forall x\ \Big[\neg\neg P(x,y)\Big] && \equiv \\
&\exists y\ \forall x\ P(x,y)
\end{aligned}$$

There is a country that every student in this class has visited

(e) **Solution:**

$$\neg\Big[\exists x\,\Big(\exists y\,\Big(x \neq y \wedge \forall z\Big[Q(x,z) \vee Q(y,z)\Big]\Big)\Big)\Big] \quad \equiv$$

$$\forall x\,\Big[\neg\Big(\exists y\,\Big(x \neq y \wedge \forall z\Big[Q(x,z) \vee Q(y,z)\Big]\Big)\Big)\Big] \quad \equiv$$

$$\forall x\,\Big[\forall y\,\neg\Big(x \neq y \wedge \forall z\Big[Q(x,z) \vee Q(y,z)\Big]\Big)\Big] \quad \equiv$$

$$\forall x\,\Big[\forall y\,\Big(x = y \vee \neg\Big(\forall z\Big[Q(x,z) \vee Q(y,z)\Big]\Big)\Big] \quad \equiv$$

$$\forall x\,\Big[\forall y\,\Big(x = y \vee \exists z\Big(\neg\Big[Q(x,z) \vee Q(y,z)\Big]\Big)\Big)\Big] \quad \equiv$$

$$\forall x\,\Big[\forall y\,\Big(x = y \vee \exists z\Big(\neg Q(x,z) \wedge \neg Q(y,z)\Big)\Big)\Big]$$

There are no two students in this class, who between them, have a friend in every country in the world.

(f) **Solution:**

$$\neg\Big[\forall x\,\forall y\,\Big(P(x,y) \to Q(x,y)\Big)\Big] \quad \equiv$$

$$\exists x\,\Big[\neg\Big(\forall y\,\Big(P(x,y) \to Q(x,y)\Big)\Big)\Big] \quad \equiv$$

$$\exists x\,\Big[\exists y\,\Big(\neg\Big(P(x,y) \to Q(x,y)\Big)\Big)\Big] \quad \equiv$$

$$\exists x\,\Big[\exists y\,\Big(\neg\Big(\neg P(x,y) \vee Q(x,y)\Big)\Big)\Big] \quad \equiv$$

$$\exists x\,\Big[\exists y\,\Big(\neg\neg P(x,y) \wedge \neg Q(x,y)\Big)\Big] \quad \equiv$$

$$\exists x\,\Big[\exists y\,\Big(P(x,y) \wedge \neg Q(x,y)\Big)\Big]$$

Somebody in this class has visited a country in which he/she doesn't have a friend.

8. Negate the following statements and transform the negation so that negation symbols immediately precede predicates. (See example in Question 7.)

(a) $\exists x \exists y \Big(P(x,y)\Big) \vee \forall x \forall y \Big(Q(x,y)\Big)$

Solution:

$$\neg\Big[\exists x\, \exists y \Big(P(x,y)\Big) \bigvee \forall x\, \forall y \Big(Q(x,y)\Big)\Big]$$

$$\equiv \neg\Big[\exists x\, \exists y \Big(P(x,y)\Big)\Big] \bigwedge \neg\Big[\forall x\, \forall y \Big(Q(x,y)\Big)\Big]$$

$$\equiv \forall x\, \forall y \Big(\neg P(x,y)\Big) \bigwedge \exists x\, \exists y \Big(\neg Q(x,y)\Big)$$

(b) $\forall x \forall y \Big(Q(x,y) \leftrightarrow Q(y,x)\Big)$

Solution:

$$\neg\Big[\forall x\, \forall y \Big(Q(x,y) \leftrightarrow Q(y,x)\Big)\Big]$$

$$\equiv \exists x\, \exists y \Big[\neg\Big(Q(x,y) \leftrightarrow Q(y,x)\Big)\Big]$$

$$\equiv \exists x\, \exists y \Big[\neg\Big[\big(Q(x,y) \rightarrow Q(y,x)\big) \bigwedge \big(Q(y,x) \rightarrow Q(x,y)\big)\Big]\Big]$$

$$\equiv \exists x\, \exists y \Big[\neg\big(Q(x,y) \rightarrow Q(y,x)\big) \bigvee \neg\big(Q(y,x) \rightarrow Q(x,y)\big)\Big]$$

$$\equiv \exists x\, \exists y \Big[\neg\big(\neg Q(x,y) \vee Q(y,x)\big) \bigvee \neg\big(\neg Q(y,x) \vee Q(x,y)\big)\Big]$$

$$\equiv \exists x\, \exists y \Big[\big(\neg\neg Q(x,y) \wedge \neg Q(y,x)\big) \bigvee \big(\neg\neg Q(y,x) \wedge \neg Q(x,y)\big)\Big]$$

$$\equiv \exists x\, \exists y \Big[\big(Q(x,y) \wedge \neg Q(y,x)\big) \bigvee \big(Q(y,x) \wedge \neg Q(x,y)\big)\Big]$$

$$\equiv \exists x\, \exists y \Big[Q(x,y) \oplus Q(y,x)\big)\Big]$$

(c) $\forall y \exists x \exists z \Big(T(x,y,z) \wedge Q(x,y)\Big)$

Solution:

$$\neg\Big[\forall y\, \exists x\, \exists z \Big(T(x,y,z) \wedge Q(x,y)\Big)\Big]$$

$$\equiv \exists y\, \forall x\, \forall z\, \neg\Big(T(x,y,z) \wedge Q(x,y)\Big)$$

$$\equiv \exists y\, \forall x\, \forall z \Big(\neg T(x,y,z) \vee \neg Q(x,y)\Big)$$