uOttawa

Faculty of science
Mathematics and Statistics

# MAT1348 : Discrete Mathematics for Computing
Spring/Summer 2017

## Test 1

Wednesday, May 24, 2017

NAME: SOLUTIONS

STUDENT NUMBER: ____________________

**Instructions:**

- Before starting the exam, please display an ID card with your **name** and **picture** on your desk. The proctor will pass during the exam to check your identity.
- This is an 80-minutes **closed-book** exam: no notes are allowed.
- The exam consists of five **long answer** questions. To receive full marks, your solution must be complete, correct, and show all relevant details.
- You may detach the last page from this exam and use it as scrap paper. It only contains a table of boolean equivalences and won't be graded.
- If you lack work space for your solutions, you may use the back pages, but please indicate it clearly.
- Cellular phones, calculators and other electronic devices are strictly **forbidden**. Those devices must be **turned off** and **stored** out of reach.
- **Good luck!**

**Question 1** (20 marks)
Let $A$ be the following formula.

$$((P \Leftrightarrow Q) \wedge (P \Rightarrow R)) \Rightarrow (Q \wedge R)$$

10 **(a)** Write down the truth table of $A$.

| | $P$ | $Q$ | $R$ | $\overbrace{P \Leftrightarrow Q}^{B}$ | $\overbrace{P \Rightarrow R}^{C}$ | $\overbrace{B \wedge C}^{D}$ | $\overbrace{Q \wedge R}^{E}$ | $\overbrace{D \Rightarrow E}^{A}$ |
|---|---|---|---|---|---|---|---|---|
| | T | T | T | T | T | T | T | T |
| | T | T | F | T | F | F | F | T |
| | T | F | T | F | T | F | F | T |
| | T | F | F | F | F | F | F | T |
| | F | T | T | F | T | F | T | T |
| | F | T | F | F | T | F | F | T |
| Line 7 → | F | F | T | T | T | T | F | F |
| Line 8 → | F | F | F | T | T | T | F | F |

4 **(b)** Is $A$ a tautology? Is it a contradiction? Is it satisfiable? Justify your answer.

It is not a tautology since the last column contains at least one 'F'. It is not a contradiction and it is satisfiable, since the last column contains at least one 'T'.

6 **(c)** Use the truth table you found in **(a)** to find a conjunctive normal form (**CNF**) for the formula $A$. *You don't need to simplify your answer.*

$$A \equiv \neg(\text{Line } 7) \wedge \neg(\text{Line } 8)$$
$$\equiv \neg(\neg P \wedge \neg Q \wedge R) \wedge \neg(\neg P \wedge \neg Q \wedge \neg R)$$
$$\equiv (P \vee Q \vee \neg R) \wedge (P \vee Q \vee R).$$

**Question 2** (20 marks)
Consider the following proposition.

> *" In order for Little Red Riding Hood not to be eaten by the wolf and for the cookies to be delivered to Grandma, it is necessary that the wolf is asleep or that the cookies are not delivered at night. "*

8 **(a)** Translate the proposition above into a formula of propositional logic.

***Hint:** Propositional variables should correspond to atomic propositions. Don't forget to state what their interpretation is!*

E = "Little Red Riding Hood is eaten by the wolf"
G = "Cookies are delivered to Grandma"
A = "The wolf is asleep"
N = "Cookies are delivered at night".

The proposition is:

$$(\neg E \wedge G) \Rightarrow (A \vee \neg N)$$

8 **(b)** Use boolean equivalences to find the **negation** of the formula that you found in **(a)**.
*Your answer should be as simple as possible.*

$$\neg((\neg E \wedge G) \Rightarrow (A \vee \neg N)) \equiv \neg(\neg(\neg E \wedge G) \vee (A \vee \neg N))$$
$$\equiv (\neg E \wedge G) \wedge \neg(A \vee \neg N)$$
$$\equiv (\neg E \wedge G) \wedge (\neg A \wedge N)$$
$$\equiv \neg E \wedge G \wedge \neg A \wedge N.$$

4 **(c)** Translate the formula you found in **(b)** back into English.

LRRH is not eaten by the wolf, the cookies are delivered to Grandma, the wolf is not asleep and the cookies are delivered at night.

**Question 3** (20 marks)

On the island of knights and knaves, three inhabitants $A$, $B$ and $C$, are being interviewed. $A$ and $B$ make the following statements:

- $A$: $B$ is a knight.
- $B$: If $A$ is a knight, then so is $C$.

The problem is to find the types of $A$, $B$ and $C$, if possible.

#9 **(a)** Translate that problem above into a satisfiability problem for propositional logic.

Let $A$ = "A is a knight"
$B$ = "B is a knight"
$C$ = "C is a knight".

Both of these formulas must be satisfied...

$A \Leftrightarrow B$ (since A says "B is a knight")

$B \Leftrightarrow (A \Rightarrow C)$ (since B says $A \Rightarrow C$).

10 **(b)** Solve the problem, using the method of your choice.

Other solutions are possible. e.g. a truth table, or a simple reasoning with words.

✓ $A \Leftrightarrow B : T$
✓ $B \Leftrightarrow (A \Rightarrow C) : T$

- $A : T$, $B : T$
  - $B : T$, ✓ $A \Rightarrow C : T$
    - $A : F$ — X
    - $C : T$ — O
  - $B : F$, $A \Rightarrow C : F$ — X
- $A : F$, $B : F$
  - $B : T$, $A \Rightarrow C : T$ — X
  - $B : F$, $A \Rightarrow C : F$
    - $A : T$, $C : F$ — X

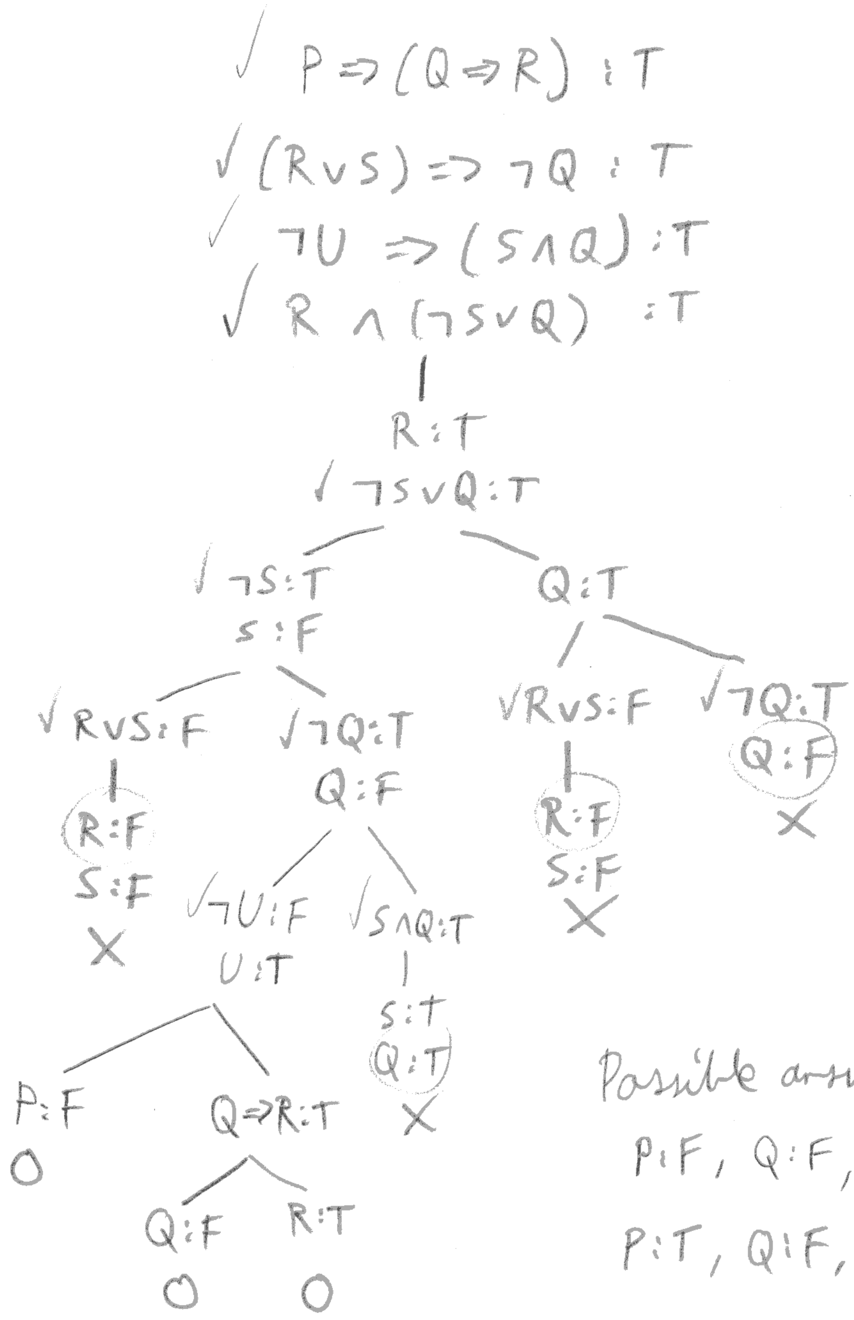
Only one open branch, for $A:T$, $B:T$, $C:T$.

$\Rightarrow$ Solution: $\boxed{\text{A, B and C are all knights.}}$

**Question 4** (20 marks)

Use a **truth tree** to determine if the following set $\mathcal{S}$ of four formulas is consistent. If you claim that it is consistent, give **one** example of an assignemt of truth values that satisfy it.

$$\mathcal{S} = \left\{ \begin{array}{c} P \Rightarrow (Q \Rightarrow R), \\ (R \vee S) \Rightarrow \neg Q, \\ \neg U \Rightarrow (S \wedge Q), \\ R \wedge (\neg S \vee Q) \end{array} \right\}$$




Possible answer: (One is enough)

P : F, Q : F, R : T, S : F, U : T

P : T, Q : F, R : T, S : F, U : T

Not all branches need to be completed, once a solution is found on a complete open branch.

**Question 5** (20 marks)

Let $A$ be the following formula.

$$(\neg(P \Leftrightarrow Q) \vee (R \vee \neg Q)) \Rightarrow \neg(S \wedge \neg P)$$

10 **(a)** Use **boolean equivalences** (*no truth table!*) to find a disjunctive normal form (**DNF**) for $A$.

$$(\neg(P \Leftrightarrow Q) \vee (R \vee \neg Q)) \Rightarrow \neg(S \wedge \neg P)$$

$$\equiv \neg(\neg(P \Leftrightarrow Q) \vee (R \vee \neg Q)) \vee \neg(S \wedge \neg P)$$

$$\equiv ((P \Leftrightarrow Q) \wedge \neg(R \vee \neg Q)) \vee \neg S \vee P$$

$$\equiv (((P \wedge Q) \vee (\neg P \wedge \neg Q)) \wedge (\neg R \wedge Q)) \vee \neg S \vee P$$

$$\equiv (P \wedge Q \wedge \neg R \wedge Q) \vee (\neg P \wedge \neg Q \wedge \neg R \wedge Q) \vee \neg S \vee P$$

$$\equiv (P \wedge Q \wedge \neg R) \vee \neg S \vee P \quad F$$

10 **(b)** Find a proposition $B$ that is logically equivalent to $A$ but uses **only** the connectives $\wedge$ and $\neg$. **Simplify** your answer as much as possible.

$$\equiv (P \wedge Q \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge Q \wedge \neg R) \vee \neg S \vee P$$

$$\equiv (P \wedge Q \wedge \neg R) \vee \cancel{(\neg P \wedge F \wedge \neg R)} \vee \neg S \vee P$$

$$\equiv (P \wedge Q \wedge \neg R) \vee \neg S \vee P$$

$$\equiv \neg(\neg(P \wedge Q \wedge \neg R) \wedge S \wedge \neg P)$$

## Table of Basic Logical Equivalences

| | Equivalence | Name |
|---|---|---|
| (1) | $P \to Q \equiv \neg P \vee Q$ | Implication Law |
| (2) | $P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$ | Biconditional Laws |
| (3) | $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$ | |
| (4) | $P \vee \neg P \equiv \mathbf{T}$ | Negation Laws |
| (5) | $P \wedge \neg P \equiv \mathbf{F}$ | |
| (6) | $P \vee \mathbf{F} \equiv P$ | Identity Laws |
| (7) | $P \wedge \mathbf{T} \equiv P$ | |
| (8) | $P \vee \mathbf{T} \equiv \mathbf{T}$ | Domination Laws |
| (9) | $P \wedge \mathbf{F} \equiv \mathbf{F}$ | |
| (10) | $P \vee P \equiv P$ | Idempotent Laws |
| (11) | $P \wedge P \equiv P$ | |
| (12) | $\neg\neg P \equiv P$ | Double negation |
| (13) | $P \vee Q \equiv Q \vee P$ | Commutative Laws |
| (14) | $P \wedge Q \equiv Q \wedge P$ | |
| (15) | $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ | Associative Laws |
| (16) | $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$ | |
| (17) | $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ | Distributive Laws |
| (18) | $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ | |
| (19) | $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ | De Morgan's Laws |
| (20) | $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$ | |