MAT 1341 D Midterm Exam II

March 25, 2019 Length: 80 minutes.

Professor: Michael Reeks

1	C
2	B
3	B
4	
5	
6	
Total	

Family name:______________________

First name:________________________

Student number:____________________

PLEASE READ THESE INSTRUCTIONS CAREFULLY:

1. You have 80 minutes to write this exam.
2. You are not allowed to consult your notes or any books. Calculators, phones, and other electronic devices are not allowed.
3. Carefully read each question and **answer all questions in the space provided for this purpose. Mark your answers to multiple choice questions in the boxes above** For questions 4 to 6, you can use the back of the pages if necessary, but do not forget to indicate it to the T.A!
4. Questions 1 to 3 are multiple choice questions, each worth 2 points. No partial credit will be awarded. You must indicate the method you used to select the correct answer; unjustified answers will not be given credit.
5. Questions 4 to 6 are long-answer questions. They are each worth 6 points. **You must justify and write your answers correctly to get all possible points.**
6. Good luck! Bonne chance!

1. Suppose that U is a subspace of $\mathbb{R}^4$ with basis $\{(1, -2, 3, 4),\ (-3, 6, -5, -16)\}$. What is the dimension of $U^\perp$?

A. 0
B. 1
C. 2
D. 3
E. 4
F. 5

The answer is C. If U is a subspace of $\mathbb{R}^n$, we always have

$$\dim U + \dim U^\perp = n.$$

Since we are given a basis for U, we can immediately tell that $\dim U = 2$. *Hence* $\dim U^\perp = 4 - 2 = 2$.

2. Suppose that

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 3 \end{pmatrix}.$$

What is the second row of A^{-1}?

A. A is not invertible
B. $(-2, -1, 1)$
C. $(-2, -2, 1)$
D. $(0, 1, 0)$
E. $(5, 4, -2)$

The answer is B. We find the inverse of A by forming the "super-augmented" matrix $(A|I_3)$, and row reducing.

$$\begin{pmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 2 & 0 & 3 & | & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & 2 & -1 & | & -2 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & -2 & -2 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & | & 3 & 3 & -1 \\ 0 & 1 & 0 & | & -2 & -1 & 1 \\ 0 & 0 & 1 & | & -2 & -2 & 1 \end{pmatrix}.$$

Hence the second row of A^{-1} is $(-2, -1, 1)$.

3. Consider the vectors

$$v_1 = (1,2,3), \quad v_2 = (2,6,8), \quad v_3 = (4,1,5), \quad v_4 = (-2,5,3),$$

each in $\mathbb{R}^3$. Which subset of these vectors forms a basis of $U = \operatorname{span}\{v_1, v_2, v_3, v_4\}$?

A. $\{v_1\}$
B. $\{v_1, v_2\}$
C. $\{v_1, v_3\}$
D. $\{v_2, v_3, v_4\}$
E. $\{v_1, v_2, v_3\}$
F. $\{v_1, v_3, v_4\}$
G. $\{v_1, v_2, v_3, v_4\}$

The answer is B. We use the column method to find a basis for U. Let

$$A = \begin{pmatrix} 1 & 2 & 4 & -2 \\ 2 & 6 & 1 & 5 \\ 3 & 8 & 5 & 3 \end{pmatrix}.$$

Then $U = \mathrm{Col}(A)$. Row reducing gives

$$A \sim \begin{pmatrix} 1 & 0 & 11 & -11 \\ 0 & 1 & -\frac{7}{2} & \frac{9}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that we don't actually have to finish the row reduction here: we just have to find where the leading 1's are. Since there are leading 1's in the first two columns of $RREF(A)$, the first two columns of A form a basis for $\mathrm{Col}(A) = V$. Thus $\{v_1, v_2\}$ is a basis for V.

4. Consider the following (3×4) matrix A:

$$A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & -1 & 4 & 3 \end{pmatrix}.$$

a) Find a basis of the column space $\mathrm{Col}(A)$ of A.

Row reducing A gives

$$A \sim \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since there are leading 1's in columns 1 and 3 in $RREF(A)$, the first and third columns of A form a basis for $\mathrm{Col}(A)$. Thus the set $\{(1,0,1),\ (2,2,4)\}$ forms a basis for $\mathrm{Col}(A)$.

b) Give a **complete** geometric description of $\mathrm{Col}(A)$.

Since $\mathrm{Col}(A)$ is a 2-dimensional subspace of $\mathbb{R}^3$, it is a plane through the origin. The normal vector is given by

$$\vec{n} = (1,0,1) \times (2,2,4) = (-2,-2,2).$$

c) Find a basis of the kernel $\mathrm{Ker}(A)$ of A.

Note that we already know $\dim \mathrm{Ker}(A)$ from part (a) and the rank-nullity theorem: since $\dim \mathrm{Ker}(A) + \mathrm{rank}(A) = \dim \mathrm{Ker}(A) + \dim \mathrm{Col}(A) = 4$, we must have $\dim \mathrm{Ker}(A) = 2$. We can use this to double check our answer later.

We can find a basis for $\mathrm{Ker}(A)$ by finding the general solution to the system $Ax = \vec{0}$. Since we've already found $RREF(A)$ in part (a), we can immediately find a spanning set for the general solution to $Ax = \vec{0}$. Let $x_2 = s$ and $x_4 = t$; then we have that the general solution is

$$\left\{ \begin{pmatrix} s+t \\ s \\ -t \\ t \end{pmatrix} |s,t \in \mathbb{R} \right\} = \mathrm{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

By a general fact from class, the spanning set for the general solution of $Ax = \vec{0}$ always forms a basis for $\mathrm{Ker}(A)$. Note that there are two vectors in the basis, which agrees with our calculation from the rank-nullity theorem earlier.

d) Calculate $\dim \mathrm{Ker}(A) + \dim \mathrm{Col}(A)$.

As noted earlier, $\dim \mathrm{Ker}(A) + \dim \mathrm{Col}(A) = 2 + 2 = 4$, which is the number of columns of A (as expected from the rank-nullity theorem).

5. Consider the subspace W of $\mathbb{R}^4$ with the basis $\{(1,0,1,0),\ (0,1,0,1),\ (0,0,0,1)\}$.

a) Find an orthogonal basis B of W.

We orthogonalize this basis using Gram-Schmidt. Let the vectors in the given basis be u_1, u_2, *and* u_3, *respectively. We have:*

1. $w_1 = u_1 = (1,0,1,0)$.

2.

$$w_2 = u_2 - \operatorname{proj}_{w_1} u_2 = (0,1,0,1) - \left(\frac{(0,1,0,1)\cdot(1,0,1,0)}{||(1,0,1,0)||^2}\right)(1,0,1,0)$$
$$= (0,1,0,1) - 0(1,0,1,0)$$
$$= (0,1,0,1).$$

Since u_1 *and* u_2 *were already orthogonal, we end up with* $w_2 = u_2$, *as well.*

3.

$$w_3 = u_3 = \operatorname{proj}_{w_1} u_3 - \operatorname{proj}_{w_2} u_3$$
$$= (0,0,0,1) - \left(\frac{(1,0,1,0)\cdot(0,0,0,1)}{||(1,0,1,0)||^2}\right)(1,0,1,0) - \left(\frac{(0,1,0,1)\cdot(0,0,0,1)}{||(0,1,0,1)||^2}\right)(0,1,0,1)$$
$$= (0,0,0,1) - 0(1,0,1,0) - (0,\frac{1}{2},0,\frac{1}{2})$$
$$= (0,-\frac{1}{2},0,\frac{1}{2}).$$

An orthogonal basis for W *is given by* $\{(1,0,1,0),\ (0,1,0,1),\ (0,-\frac{1}{2},0,\frac{1}{2})\}$.

b) Find the best approximation in W of the vector $v = (0,1,-1,1)$.

The best approximation to $(0,1,-1,1)$ *in* W *is given by the projection* $\operatorname{proj}_W(0,1,-1,1)$. *We have*

$$\operatorname{proj}_W(0,1,-1,1) = \operatorname{proj}_{w_1}(0,1,-1,1) + \operatorname{proj}_{w_2}(0,1,-1,1) + \operatorname{proj}_{w_3}(0,1,-1,1)$$
$$= \left(\frac{(0,1,-1,1)\cdot(1,0,1,0)}{||(1,0,1,0)||^2}\right)(1,0,1,0) + \left(\frac{(0,1,-1,1)\cdot(0,1,0,1)}{||(0,1,0,1)||^2}\right)(0,1,0,1)$$
$$+ \left(\frac{(0,1,-1,1)\cdot(0,-\frac{1}{2},0,\frac{1}{2})}{||(0,-\frac{1}{2},0,\frac{1}{2})||}\right)(0,-\frac{1}{2},0,\frac{1}{2})$$
$$= (-\frac{1}{2},0,-\frac{1}{2},0) + (0,1,0,1) + (0,0,0,0)$$
$$= (-\frac{1}{2},1,-\frac{1}{2},1).$$

c) Extend the orthogonal basis B from part (a) to a basis of $\mathbb{R}^4$.
Write the vectors from part (a) as the rows of a matrix and row reduce:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The only column without a leading 1 is column 3. Hence adding the vector $(0,0,1,0)$ *extends the basis from part (a) to a basis for* $\mathbb{R}^4$*. The full basis is*

$$\{(1,0,1,0),\ (0,1,0,1),\ (0,-\frac{1}{2},0,\frac{1}{2}),(0,0,1,0)\}.$$

6(a). In each case, indicate in the corresponding box if the statement below is (always) true or can be false.

- If you think that the statement may be wrong, give an explicit example that it is false.
- If you think that the statement is (always) true, you must justify it with a clear explanation.

I. If A is an invertible matrix and $AB = 0$, then $B = 0$.

We can multiply both sides of the equation on the left by A^{-1} to obtain

$$A^{-1}AB = A^{-1}0$$

$$\Rightarrow I_n B = 0$$

$$\Rightarrow B = 0.$$

ANSWER *TRUE*

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II. The columns of a (4×3) matrix are always linearly dependent (LD).

The columns of a 4×3 matrix are vectors in $\mathbb{R}^4$, and there are 3 of them. It is certainly possible to find 3 linearly independent vectors in $\mathbb{R}^4$, since it has dimension 4: consider, for instance, the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that if we transposed the matrix (so that it was 3×4 rather than 4×3), the answer would be true: it's impossible for a set of 4 vectors in $\mathbb{R}^3$ to be linearly independent.

ANSWER *FALSE*

6(b). Let A be an $(n \times n)$ matrix with real entries. Give three equivalent statements to the statement

"A is an invertible matrix"

following these specifications:

(i) One which gives a condition on the **rank of** A

A is invertible if and only if $\operatorname{rank}(A) = n$*. We can see this from the algorithm for finding A^{-1}: if A^{-1} exists, we need $RREF(A)$ to be the $n \times n$ identity matrix, so there must be a leading 1 in each row and column. Hence* $\operatorname{rank}(A) = n$*.*

(ii) One gives a condition on the **reduced row echelon form of** A

As noted above, we must have that $RREF(A) = I_n$. To find the inverse of an $n \times n$ matrix, we form the super-augmented matrix $(A|I_n)$. If A^{-1} exists, we need the row reduced form of this augmented matrix to have the form $(I_n|A^{-1})$. Note that this also follows from the rank condition in part (i).

(iii) One which gives a condition on the **columns of** A

A matrix A is invertible if and only if the columns of A are linearly independent. This follows from the previous two parts: if $RREF(A) = I_n$, then the method for finding a basis for the column space of A would give that the columns of A already form a basis for $\operatorname{Col}(A)$*, i.e. that they are linearly independent. Also,* $\operatorname{rank}(A) = \dim \operatorname{Col}(A)$*; since the columns of A form a spanning set for* $\operatorname{Col}(A)$ *by definition, and there are* $\dim \operatorname{Col}(A)$ *many of them, they must be linearly independent.*