December 2012 - Final Exam Solutions

(1)(a)(i) Find limits

$$\lim_{x \to 1} f(x)$$

$$= \lim_{x \to 1} \frac{3 - x^2}{(2x^3 - x^2 + 9)}$$

$$= \frac{3 - 1^2}{2(1)^3 - (1)^2 + 9}$$

$$= \frac{2}{10}$$

$$\therefore \lim_{x \to 1} \frac{3 - x^2}{2x^3 - x^2 + 9} = \frac{1}{5}$$

(ii) $\lim_{x \to \infty} f(x)$

$$= \lim_{x \to \infty} \frac{3 - x^2}{2x^3 - x^2 + 9} = \lim_{x \to \infty} \frac{3}{2x^3} = 0$$

$$\therefore \lim_{x \to \infty} \frac{3 - x^2}{2x^3 - x^2 + 9} = 0$$

(b) $\lim_{x \to 3} \sqrt{g(x) - h(x)}$ where $\lim_{x \to 3} g(x) = 4$

$\lim_{x \to 3} h(x) = -5$

<u>Sol</u>

$$\lim_{x \to 3} \sqrt{g(x) - h(x)}$$

$$= \sqrt{\lim_{x \to 3} [g(x) - h(x)]}$$

$$= \sqrt{\lim_{x \to 3} g(x) - \lim_{x \to 3} h(x)}$$

$= \sqrt{4-(-5)}$

$= 3$

$\therefore \lim_{x \to 3} \sqrt{g(x) - h(x)} = 3$

(c) False.

Counter-example is the constant function $g(x) = 5$, where $\lim_{x \to 4} g(x) = 5$ and $\lim_{x \to 3} g(x) = 5$.

(2) (a) $g'(x) = \frac{d}{dx} g(x) = \frac{d}{dx}(-3x^4 + 2x^2 - \pi)$

$= \frac{d}{dx}(-3x^4) + \frac{d}{dx}(2x^2) - \frac{d}{dx}(\pi)$

$= -12x^3 + 4x - 0$

$\therefore g'(x) = -12x^3 + 4x$

(b) $f'(x) = \frac{d}{dx}(f(x)) = \frac{d}{dx}\left((\ln x + x)(2x^2 - 5)\right)$, use ~~chain~~ Product rule,

$= \frac{d}{dx}(\ln x + x)(2x^2-5) + (\ln x + x)\frac{d}{dx}(2x^2-5)$

$= \left(\frac{1}{x} + 1\right)(2x^2-5) + (\ln x + x)(4x)$

Expand...

$$= 2x - \frac{5}{x} + 2x^2 - 5 + 4x\,Ln\,x + 4x^2$$

$$= 6x^2 + 2x - \frac{5}{x} + 4x\ln(x) - 5$$

(c) $y' = \frac{d}{dx}\left(\frac{e^x - x}{x^2 - 2x}\right)$

use Quotient Rule...

$$= \frac{\frac{d}{dx}(e^x - x)(x^2 - 2x) - (e^x - x)\frac{d}{dx}(x^2 - 2x)}{(x^2 - 2x)^2}$$

$$= \frac{(e^x - 1)(x^2 - 2x) - (e^x - x)(2x - 2)}{(x^2 - 2x)^2}$$

$$= \frac{x^2e^x - 2xe^x - x^2 + 2x - (2e^x x - 2e^x - 2x^2 + 2x)}{x^2(x-2)^2}$$

$$\therefore y' = \frac{x^2 + e^x(x^2 - 4x + 2)}{(x-2)^2 x^2}$$

(d) $y = \sqrt[3]{x^5 - 7} = (x^5 - 7)^{1/3}$

Chain rule:

$$\frac{dy}{dx} = \frac{1}{3}(x^5 - 7)^{-2/3} \cdot (5x^4)$$

$$\therefore y' = \frac{5}{3} \cdot \frac{x^4}{\sqrt[3]{(x^5 - 7)^2}}$$

3 continued

(2) (e) Find y' if $e^y = y^3 - 2x$

$$\frac{d}{dx} e^y = \frac{d}{dx}\left(y^3 - 2x\right)$$ ... implicit differentiation.

$$y' e^y = 3y^2 y' - 2$$

solve for y':

$$y'(e^y - 3y^2) = -2$$

$$\therefore \; y' = \frac{-2}{e^y - 3y^2}$$

(3) $0.03x + 4p = 30$

(a) Demand as function of price:

$$0.03x = 30 - 4p$$
$$x = \frac{30 - 4p}{0.03}$$
$$\therefore x = 1000 - \frac{400}{3}p = f(p)$$

(b) Revenue, $R(p) = xp = p\,f(p)$

$$= p\left(1000 - \frac{400}{3}p\right)$$
$$= 1000p - \frac{400}{3}p^2$$
$$\therefore R(p) = 1000p - \frac{400}{3}p^2$$

(c) Elasticity of Demand, $E(p)$

$$E(p) = -p\,\frac{f'(p)}{f(p)}$$

(i) $f'(p) = \frac{d\,f(p)}{dp} = \frac{d}{dp}(1000 - 4p) = -4$

(ii) $E(p) = \frac{(-p)(-4)}{1000 - \frac{400}{3}p}$

$$= \frac{4p}{1000 - \frac{400}{3}p}$$

$$\therefore E(p) = \frac{12p}{3000 - 400p}$$

(4) $C(x) = 1000 + 25x - 0.1x^2$

(a) Find $\bar{C}(x)$ and $\bar{C}'(x)$

(i) $\bar{C}(x) = \dfrac{C(x)}{x} = \dfrac{1000 + 25x - 0.1x^2}{x}$

$\therefore \bar{C}(x) = \dfrac{1000}{x} + 25 - 0.1x$, average cost

(ii) $\bar{C}'(x) = \dfrac{d}{dx}\left(\bar{C}(x)\right) = \dfrac{d}{dx}\left(\dfrac{1000}{x} + 25 - 0.1x\right)$

$\therefore \bar{C}'(x) = -\dfrac{1000}{x^2} - 0.1$, Marginal average cost

(b) Find $\bar{C}(10)$ and $\bar{C}'(10)$, and interpret.

(i) $\bar{C}(10) = \dfrac{1000}{10} + 25 - 0.1(10)$

$= 100 + 25 - 1$

$= 124$

$\therefore \bar{C}(10) = \$124$

(ii) $\bar{C}'(10) = -\dfrac{1000}{10^2} - 0.1$

$\therefore \bar{C}'(10) = \$ -10.10$

$\therefore$ At a production level of 10 bits per day, the average cost of producing a bit is \$124.

The cost decreases at a rate of \$10.10 per bit.

(4) (c) Estimate the average cost per bit at a production level of 11 bits per day.

Sol:

If production is increased by 1 bit, then the average cost per bit will decrease by approximately \$10.10.
So, the average cost per bit at a production level of 11 bits per day is approximately \$124 - \$10.10 = \$113.90.

(5) Find dy for $y = \sqrt{x} + 3$ and evaluate dy.

(a) $$\frac{dy}{dx} = \frac{d}{dx}(x^{1/2} + 3) = \frac{d(x^{1/2})}{dx} + \frac{d(3)}{dx}$$

$$= \frac{1}{2}x^{-1/2} + 0 = \frac{1}{2\sqrt{x}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \rightarrow dy = \frac{1}{2\sqrt{x}}\,dx$$

If $x = 4$ and $dx = 0.1$:

$$dy = \frac{1}{2\sqrt{4}}(0.1)$$

$$= \frac{0.1}{4} = \frac{1}{40}$$

$\therefore$ With $x = 4$ and $dx = 0.1$, $dy = \frac{1}{40}$.

(5) b. $x = 9$, $dx = 0.12$

$$dy = \frac{1}{2\sqrt{9}}(0.12) = \frac{1}{2\cdot 3}\cdot(0.12) = 0.02.$$

$\therefore$ with $x = 9$ and $dx = 0.12$, $dy = 0.02$

(6) Compute Integrals

(a) $\int e^{-3x}\,dx$, let $u = 3x$
$du = 3\,dx$
$\Rightarrow dx = \frac{du}{3}$

so...

$$= \int e^{-u}\,du$$

$$= -e^{-u} + C$$

$$= -e^{-3x} + C$$

$$\therefore \int e^{-3x}\,dx = -e^{-3x} + C$$

(6) (b) $\int (4x^3 - 7x^6)\, dx$

$= 4\int x^3\, dx - 7\int x^6\, dx$

$= 4\frac{x^4}{4} - 7\frac{x^7}{7} + C$

$= x^4 - x^7 + C$

$\therefore \int (4x^3 - 7x^6)\, dx = x^4 - x^7 + C$

(c) $\int (x+9)^{-8}\, dx$

$\Downarrow$

let $u = x+9$

$du = dx$

$\Downarrow$

$= \int u^{-8}\, du$

$= -\frac{u^{-7}}{7} + C = -\frac{(x+9)^{-7}}{7} + C$

$\therefore \int (x+9)^{-8}\, dx = -\frac{1}{7}(x+9)^{-7} + C$

(6)(d) $\int (ex^5 - x^2)\, dx$

$$= \int e x^5 \, dx - \int x^2 \, dx$$

$\uparrow$ (i) $\quad\quad \uparrow$ (ii)

Some constant

(i) $\int e x^5 \, dx = e \int x^5 \, dx = e\dfrac{x^6}{6} + C_1$

(ii) $\int x^2 \, dx = \dfrac{x^3}{3} + C_2$ Some other constant

Now, (i) - (ii) ;

Some constant called C.

$$= e\frac{x^6}{6} - \frac{x^3}{3} + (C_1 + C_2)$$

$$= e\frac{x^6}{6} - \frac{x^3}{3} + C$$

$$= \frac{x^3}{3}\left(\frac{ex^3}{2} - 1\right) + C$$

$$\therefore \int (ex^5 - x^2)\, dx = \frac{x^3}{3}\left(\frac{ex^3}{2} - 1\right) + C$$

(6) (e) $\int \frac{x^2}{7-x^3}\,dx$

$\Downarrow$

let $u = 7 - x^3$

$du = -3x^2\,dx$

$\Leftrightarrow x^2 dx = -\frac{du}{3}$

$\Downarrow$

so...

$$= \frac{-1}{3}\int \frac{du}{u} = \frac{-1}{3}\,\mathrm{Ln}(u) + C = \frac{-1}{3}\,\mathrm{Ln}(7-x^3) + C$$

(f) $\int x(x^2-5)^{-6}\,dx$

$\Downarrow$

let $u = x^2 - 5$

$du = 2x\,dx$

$\Leftrightarrow x\,dx = \frac{du}{2}$

$$= \frac{1}{2}\int u^{-6}\,du = \frac{-1}{2}\,\frac{u^{-5}}{5} + C = \frac{-1}{10}(x^2-5)^{-5} + C$$

$$\therefore \int x(x^2-5)^{-6}\,dx = \frac{-1}{10}(x^2-5)^{-5} + C$$

(7) Find absolute max. and absolute min. value of
$f(x) = x^3 - 12x$ on $[-3, 3]$.

(i) $f(x)$ is continuous ✓

(ii) Find critical values x, $f'(x) = 0$

$f'(x) = 3x^2 - 12$
$0 = 3x^2 - 12$
$12 = 3x^2$
$x^2 = 4$
$x = 2, -2$

We have critical values $x = -2$ and $x = 2$.
And we have end points $x = -3$ and $x = 3$.

Lets test these values:

x	$f(x)$	$f(x) = x^3 - 12x$
-2	16	← Absolute Max.
2	-16	← Absolute Min.
-3	9	
3	-9	

∴ In the interval $[-3, 3]$ for $f(x) = x^3 - 12x$, the absolute maximum is $(-2, 16)$ and the absolute minimum is $(2, -16)$.

(8) Yes, there is such a function!

$$f(x)\begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

so $\lim_{x \to 0^+} f(x) = -1$ and $\lim_{x \to 0^-} f(x) = 1$, which makes it discontinuous.

However if we square it:

$$f(x)\begin{cases} 1^2 = 1, & x \geq 0 \\ (-1)^2 = 1, & x < 0 \end{cases}$$

So $\lim_{x \to 0^+} f(x) = 1$ and $\lim_{x \to 0^-} f(x) = 1$, is now continuous.

(9) Find all pertinent information for $h(x) = \frac{2x-1}{x^2}$ and sketch graph.

(i) Analyze $h(x)$

Domain: All real x, except $x=0$.

X-intercept: $0 = \frac{2x-1}{x^2} \rightarrow x = \frac{1}{2}$

y-intercept: Since $x=0$ is not in the domain, there is no y-intercept.

Horizontal Asymptote: $\lim_{x\to\infty} \frac{2x-1}{x^2} = \lim_{x\to\infty} \frac{2}{x} = 0$

$\therefore y=0$

Vertical Asymptote:

Denominator is 0 at $x=0$ where the numerator is not 0.
So y axis (the line $x=0$) is the vertical asymptote.

(ii) Analyze $h'(x)$

Quotient Rule.

$$h'(x) = \frac{2x^2 - (2x-1)(2x)}{(x^2)^2}$$

$$= \frac{2x^2 - 4x^2 + 2x}{x^4}$$

$$= \frac{-2x^2 + 2x}{x^4}$$

$$\therefore h'(x) = \frac{2(1-x)}{x^3}$$

Now find critical values x where $h'(x) = 0$.

$h'(x) = 0$

$0 = \dfrac{2(1-x)}{x^3}$

$0 = 2(1-x)$

$\therefore x = 1$ is a critical value.

Partition numbers for $h'(x) = 0$: $x = 0$, $x = 1$.

Sign chart for $h'(x)$:

$h'(x)$: $(-\infty, 0)$ $\circ$ 0 $(0,1)$ $\bullet$ 1 $(1,\infty)$ $\to x$

0 ← not defined

$h'(x) = \dfrac{2(1-x)}{x^3}$

interval	$h'(x)$
$(-\infty, 0)$	negative
$(0,1)$	positive
$(1,\infty)$	negative

So $h(x)$ is decreasing on $(-\infty, 0)$
increasing on $(0,1)$
decreasing on $(1,\infty)$.

(iii) Analyze $h''(x)$

Quotient Rule:

$$h''(x) = \frac{-2x^3 - (2-2x)(3x^2)}{(x^3)^2}$$

$$= \frac{-2x^3 - (2-2x)(3x^2)}{x^6}$$

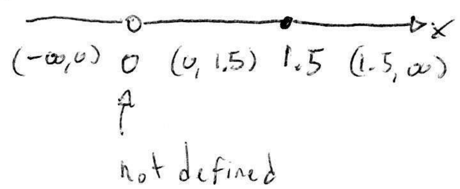
$$\therefore h''(x) = \frac{2(2x-3)}{x^4}$$

Inflection points at $h''(x) = 0$.

$0 = \dfrac{2(2x-3)}{x^4}$, $x = \dfrac{3}{2}$ is an inflection point.

Partition numbers for $h''(x)$: $x=0$, $x=1.5$

$h''(x)$



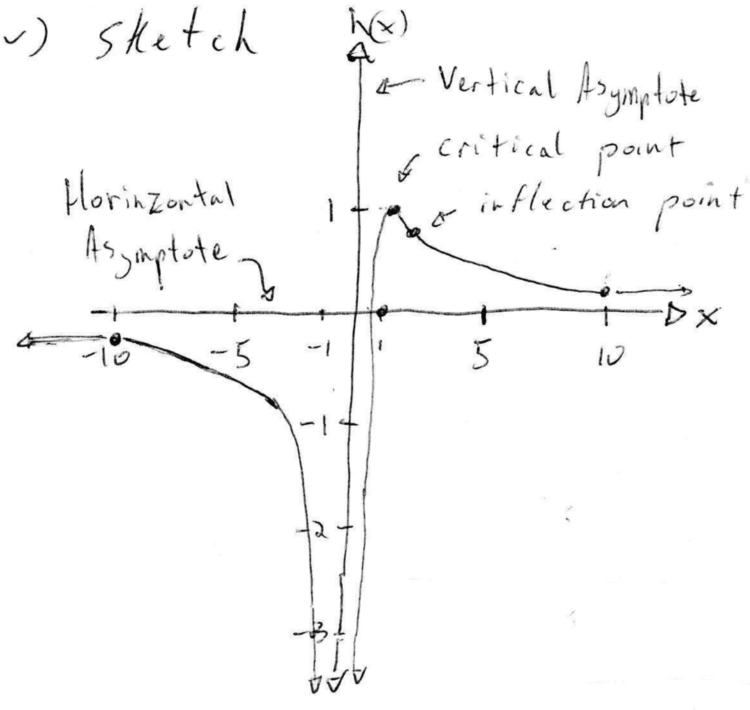
$$h''(x) = \frac{2(2x-3)}{x^4}$$

interval	$h''(x)$
$(-\infty, 0)$	negative
$(0, 1.5)$	negative
$(1.5, \infty)$	positive

So $g(x)$ is:

Concave downward on $(-\infty, 0)$
Concave downward on $(0, 1.5)$
Concave upward on $(1.5, \infty)$
with an inflection point at $x = 1.5$.

(iv) Sketch



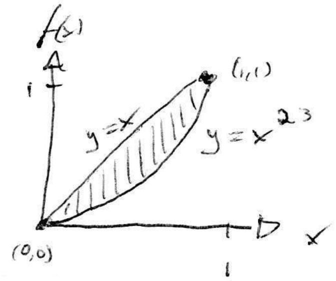
Test points

x	g(x)
-10	-0.21
-1	-3
0.5	0
1	1
1.5	0.89
10	0.19

(10) Gini index for Lorenz curve $f(x) = 2\int_0^1 (x - f(x))\,dx$.

The Lorenz curve for a small country is $x^{2.3}$.

Graph the curve; find the Gini index.



$$2\int_0^1 (x - f(x))\,dx = 2\int_0^1 (x - x^{2.3})\,dx = 2\left[\frac{x^2}{2} - \frac{x^{3.3}}{3.3}\right]\Big|_0^1$$

$$= 2\left[\left(\frac{1}{2} - \frac{1}{3.3}\right) - (0 - 0)\right]$$

$$= 1 - \frac{2}{3.3} \approx 0.394$$

END