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1. a) $V(r) = \frac{4}{3}\pi r^3$ given: $r = 5$ cm, $dr = 0.1$ cm

$$\frac{dV}{dr} = V' = 3 \cdot \frac{4}{3} \cdot \pi \cdot r^2$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$dV = 4\pi r^2 \cdot dr$$

$$dV = 4\pi (5)^2 (0.1)$$

$$dV = 10\pi$$

$$dV \approx \underline{31.4 \text{ cm}^3}$$

b) Consider the function $f(x) = \underline{|x|}$

it is continuous at $x = 0$ because:

$$\left.\begin{aligned} \lim_{x \to 0^-} |x| &= \lim_{x \to 0^-} -x = 0 \\ \lim_{x \to 0^+} |x| &= \lim_{x \to 0} x = 0 \end{aligned}\right\}$$

(1) the limit of $f(x)$ exists at $x = 0$

$$f(0) = |0| = 0$$

(2) and the value of the limit equals the value of the function at $x = 0$

but the graph of $f(x) = |x|$ has a sharp corner at $x = 0$ and thus is not differentiable.

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2. a) $f(x) = 4 - 6x^{10} - 4x^3$ power rule

$f'(x) = \boxed{-60x^9 - 12x^2}$

b) $f(x) = \dfrac{x^2+5x}{e^x - 7}$ f(x) cannot be simplified $\Rightarrow$ quotient rule

$u = x^2 + 5x$ $\quad v = e^x - 7$

$u' = 2x+5$ $\quad v' = e^x$

$$f'(x) = \frac{v \cdot u' - u \cdot v'}{v^2}$$

$$f'(x) = \boxed{\frac{(e^x-7)(2x+5) - (x^2+5x)(e^x)}{(e^x-7)^2}}$$

c) $y = \ln(x^2+3x)^2$

assume that it's : $\ln\left[(x^2+3x)^2\right]$

and not : $\left[\ln(x^2+3x)\right]^2$

$y = \ln(x^2+3x)^2$. chain rule

$u = (x^2+3x)^2$

$u' = 2(x^2+3x)(2x+3)$

$y = \ln(u)$

$$\frac{dy}{dx} = \frac{1}{u} \cdot u' = \boxed{\frac{1}{(x^2+3x)^2} \cdot 2(x^2+3x)(2x+3)}$$

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2. d) $y = \sqrt[5]{x+5} = (x+5)^{1/5}$

$$\frac{dy}{dx} = \frac{1}{5}(x+5)^{-1/5}\cdot(1)$$

$$\frac{dy}{dx} = \boxed{\frac{1}{5\cdot\sqrt[5]{x+5}}}$$

e) $xy = e^y - 2$ assume y is a function of x and use implicit differentiation

$$\frac{d}{dx}(xy) = \frac{d}{dx}\left(e^y - 2\right)$$

[II] pause: $\frac{d}{dx}(x\cdot y) \to$ product rule

$u = x \qquad v = y$

$u' = 1 \qquad v' = y'$

$$\frac{d}{dx}(xy) = u\cdot v' + u'\cdot v = \boxed{x\cdot y' + 1\cdot y}$$

$$\frac{d}{dx}(e^y - 2) = \boxed{e^y\cdot y'} \quad \text{(chain rule)}$$

[▷] resume:

$$\frac{d}{dx}(xy) = \frac{d}{dx}(e^y - 2)$$

$$x\cdot y' + y = e^y\cdot y'$$

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2. e) $x \cdot y' + y = e^y \cdot y'$ solve for y'

$$x \cdot y' - e^y \cdot y' = -y$$

$$y'(x - e^y) = -y$$

$$\boxed{y' = \frac{-y}{x - e^y}}$$

3. given $x = 1000 - 20p$ (demand as a function of price)

a) $x = 1000 - 20p$

$$x - 1000 = -20p$$

$$\boxed{p = -\frac{1}{20}x + 50}$$

since price cannot be negative:

$$p > 0$$

$$-\frac{1}{20}x + 50 > 0$$

$$-\frac{1}{20}x > -50$$

$$x < 1000$$

quantity can also not be negative, therefore the domain is:

$0 < x < 1000$, x is a <u>NATURAL</u> number.

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3\. b) revenue = quantity · price

$R(x) = x \cdot p$

$R(x) = x \cdot \left(-\frac{1}{20}x + 50\right)$

$\boxed{R(x) = -\frac{1}{20}x^2 + 50x}$, domain is same as $p(x)$ that is $0 < x < 1000$, x is a NATURAL number.

c) marginal revenue function $= R'(x)$

$R'(x) = -\frac{1}{10}x + 50$

$R'(400) = -\frac{1}{10}(400) + 50 = \underline{\underline{10}}$

at a production level of 400 steam irons, our revenue would increase by \$10 for each additional steam iron we produce.

d) $R'(650) = -\frac{1}{10}(650) + 50 = \underline{\underline{-15}}$

at a production level of 650 steam irons, our revenue would decrease by \$15 for each additional steam iron we produce

NOTE: in both parts (c) and (d), the given quantities (400 and 650) were in the domain of $R(x)$.

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4. given: current price \$4

$$x = f(p) = 7000 - 50p$$

to determine if revenue will increase or decrease with a change in price, let's evaluate the elasticity of demand at the current price.

$$E(p) = -\frac{p\,f'(p)}{f(p)}$$

$$f(p) = 7000 - 50p$$

$$f'(p) = -50$$

$$E(p) = \frac{-p(-50)}{7000 - 50p} = \frac{50p}{7000 - 50p}$$

at a price of \$4: $E(4) = \dfrac{50(4)}{7000 - 50(4)} = \underline{\underline{0.0294}} < 1$

since the elasticity of demand is less than 1, the demand is inelastic, and therefore an <u>increase</u> in price will <u>increase</u> revenue.

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5\. a) $\int e^{5x}\,dx$

let $\boxed{u = 5x}$

$\frac{du}{dx} = 5$

$du = 5\,dx \Rightarrow \boxed{dx = \frac{1}{5}\,du}$

$= \int e^{u}\left(\frac{1}{5}\,du\right)$

$= \frac{1}{5}\int e^{u}\,du = \frac{1}{5}e^{u} + C = \boxed{\frac{1}{5}e^{5x} + C}$

b) $\int \frac{x}{\sqrt{x-7}}\,dx = \int x(x-7)^{-1/2}\,dx$

let $\boxed{u = x-7}$ then $\boxed{x = u+7}$

and $\frac{du}{dx} = 1 \Rightarrow \boxed{du = dx}$

$\int x(x-7)^{-1/2}\,dx$

$= \int (u+7)(u^{-1/2})\,du$

$= \int u^{1/2} + 7u^{-1/2}\,du$

$= \frac{2}{3}u^{3/2} + 7\left(\frac{2}{1}\right)u^{1/2} + C$

$= \boxed{\frac{2}{3}(x-7)^{3/2} + 14(x-7)^{1/2} + C}$

c) $\int (3x^2 + 5x)\,dx = \boxed{x^3 + \frac{5}{2}x^2 + C}$

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5. d) $\int \frac{x^2}{4+x^3}\, dx$ let $u = 4+x^3$

$\frac{du}{dx} = 3x^2$

$\frac{1}{3}\, du = x^2\, dx$

$= \int \frac{\frac{1}{3}\cdot du}{u}$

$= \frac{1}{3} \int \frac{1}{u}\, du = \frac{1}{3}\ln(u) + C = \boxed{\frac{1}{3}\ln(4+x^3) + C}$

e) $\int \left((x^2+1)^{12} \cdot x \right) dx$ let $u = x^2+1$

$\frac{du}{dx} = 2x$

$\frac{1}{2}\, du = x \cdot dx$

$= \int u^{12} \left(\frac{1}{2}\, du\right)$

$= \frac{1}{2} \int u^{12}\, du = \frac{1}{2}\left(\frac{1}{13}\right) u^{13} + C = \boxed{\frac{1}{26}(x^2+1)^{13} + C}$

6. $f(x) = x^2 - x$ $g(x) = 2x$ $-2 \leq x \leq 3$

find where graphs intersect:

$f(x) = g(x)$

$x^2 - x = 2x$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0,\ x = 3$

now have to find which graph is on top and which is on bottom

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6.

interval	$-2 < x < 0$	$0 < x < 3$
test value	$x = -1$	$x = 1$
$f(x)$	2	0
$g(x)$	-2	2
	$f(x) > g(x)$	$g(x) > f(x)$

$\therefore$ $f(x)$ is on top from -2 to 0
$g(x)$ is on top from 0 to 3

$$\text{Area} = \int (\text{top curve} - \text{bottom curve})\,dx$$

$$A = \int_{-2}^{0} f(x) - g(x)\,dx + \int_{0}^{3} g(x) - f(x)\,dx$$

$$f(x) - g(x) = x^2 - x - 2x = x^2 - 3x$$
$$g(x) - f(x) = 2x - (x^2 - x) = 2x - x^2 + x = -x^2 + 3x$$

$$A = \int_{-2}^{0} x^2 - 3x\,dx + \int_{0}^{3} -x^2 + 3x\,dx$$

$$A = \left(\frac{1}{3}x^3 - \frac{3}{2}x^2\right|_{-2}^{0} + \left(-\frac{1}{3}x^3 + \frac{3}{2}x^2\right|_{0}^{3}$$

$$A = \left(\cancel{\frac{1}{3}(0)^3}^{\,0} - \cancel{\frac{3}{2}(0)^2}^{\,0} - \left(\frac{1}{3}(-2)^3 - \frac{3}{2}(-2)^2\right)\right) + \left(-\frac{1}{3}(3)^3 + \frac{3}{2}(3)^2 - \left(\cancel{-\frac{1}{3}(0)^2}^{\,0} + \cancel{\frac{3}{2}(0)^2}^{\,0}\right)\right)$$

$$A = \frac{8}{3} + 6 + (-9) + \frac{27}{2}$$

$$A = \underline{\underline{\frac{79}{6}}} \approx \underline{\underline{13.167}} \text{ square units}$$

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7. given : $\frac{P}{T} = k$ P, T are both functions of time, t
k is constant

$$\frac{dT}{dt} = 3 \text{ Kelvin / hour}$$

$$T = 250 \text{ Kelvin}$$

$$P = 500 \text{ psi}$$

at the given P and T $\Rightarrow$ $k = \frac{P}{T} = \frac{500}{250} = 2$

therefore the equation that relates pressure and temperature is:

$$\frac{P}{T} = 2 \quad \text{or} \quad P = 2T$$

to find rate of change of pressure, differentiate both sides w.r.t. time, t:

$$\frac{d}{dt}(P) = \frac{d}{dt}(2T)$$

$$\frac{dP}{dt} = 2 \cdot \frac{dT}{dt}$$

$$\frac{dP}{dt} = 2(3)$$

$$\frac{dP}{dt} = \underline{\underline{6}} \text{ psi per hour}$$

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8. a) $\int_0^5 (t^2 - 4)\,dt = \frac{1}{3}t^3 - 4t \Big|_0^5$

$= \left(\frac{1}{3}(5)^3 - 4(5)\right) - \left(\frac{1}{3}(0)^3 - 4(0)\right)$

$= \frac{1}{3}(125) - 20$

$= \underline{\underline{\frac{65}{3}}}$

b) ERROR : integrand should be $h \cdot e^{h^2}$

$\int_2^3 h e^{h^2}\, dh$

let $u = h^2$

$\frac{du}{dh} = 2h$

$\frac{1}{2} du = h \cdot dh$

when $h = 2$ $\Rightarrow u = 2^2 = 4$

when $h = 3$ $\Rightarrow u = 3^2 = 9$

$= \int_4^9 \frac{1}{2} e^u \cdot du = \frac{1}{2}\int_4^9 e^u\, du = \frac{1}{2} e^u \Big|_4^9$

$= \frac{1}{2}e^9 - \frac{1}{2}e^4$

$= \underline{\underline{\frac{1}{2}e^4\left(e^5 - 1\right)}}$ $\approx 4024{,}243$

9. a) (i) $\lim_{x \to -3} \frac{x^2 - 3x + 2}{x - 1} = \frac{(-3)^2 - 3(-3) + 2}{-3 - 1} = \frac{20}{-4} = \underline{\underline{-5}}$

(ii) $\lim_{x \to 5} \frac{x^2 - 16}{x - 5} = \lim_{x \to 5} \frac{(x-4)(x+4)}{x - 5}$ no common factors so we have to investigate right & left limits

$\lim_{x \to 5^-} \frac{(x-4)(x+4)}{x-5} = -\infty$, $\lim_{x \to 5^+} \frac{(x-4)(x+4)}{x-5} = \infty$

$\therefore$ the given limit $\underline{\underline{\text{does not}}}$ exist.

(iii) $\lim_{x \to \infty} \frac{-5x^7 + 3x^2 + 2}{4 - x^2}$

$= \lim_{x \to \infty} \frac{-\frac{5x^7}{x^2} + \frac{3x^2}{x^2} + \frac{2}{x^2}}{\frac{4}{x^2} - \frac{x^2}{x^2}}$

$= \lim_{x \to \infty} \frac{-5x^5 + 3 + \frac{2}{x^2}}{\frac{4}{x^2} - 1}$

$= \frac{\lim_{x\to\infty} -5x^5 + \cancel{\lim_{x\to\infty}} 3 + \cancel{\lim_{x\to\infty} \frac{2}{x^2}}^{0}}{\cancel{\lim_{x\to\infty} \frac{4}{x^2}}^{0} - \cancel{\lim_{x\to\infty} 1}\ 1}$

$= \lim_{x\to\infty} \frac{-5x^5 + 3}{-1} = \lim_{x\to\infty} 5x^5 - 3 = \underline{\underline{\infty}}$

b) given: $\lim_{x\to 3} f(x) = -5$, $\lim_{x\to 3} g(x) = 4$

(i) $\lim_{x\to 3} (-3g(x)) = -3 \cdot \lim_{x\to 3} g(x) = -3(4) = \underline{\underline{-12}}$

(ii) $\lim_{x\to 3} \sqrt{g(x)} = \sqrt{\lim_{x\to 3} g(x)} = \sqrt{4} = \underline{\underline{2}}$

(iii) $\lim_{x\to 3} \frac{g(x)}{2 \cdot f(x)} = \frac{1}{2} \cdot \frac{\lim_{x\to 3} g(x)}{\lim_{x\to 3} f(x)} = \frac{1}{2} \cdot \frac{4}{-5} = \underline{\underline{-\frac{2}{5}}}$

(iv) $\lim_{h\to 0} \frac{(x-h)^2 - x^2}{h} = \lim_{h\to 0} \frac{\cancel{x^2} - 2xh + h^2 - \cancel{x^2}}{h}$

$= \lim_{h\to 0} \frac{-2xh + h^2}{h}$

$= \lim_{h\to 0} \frac{h(-2x + h)}{h}$

$= \lim_{h\to 0} -2x + h = \underline{\underline{-2x}}$

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10. $f(x) = x^4 - 2x^3$

1) y- intercept at $x=0 \Rightarrow f(0) = (0)^4 - 2(0)^3 = 0 \Rightarrow (0,0)$

2) x-intercepts at $y = f(x) = 0 \Rightarrow x^4 - 2x^3 = 0$

$x^3(x-2) = 0$

$x = 0, \quad x = 2 \Rightarrow (0,0)$
$(2,0)$

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3) critical points at $f'(x) = 0 \Rightarrow f'(x) = 4x^3 - 6x^2$

$4x^3 - 6x^2 = 0$
$2x^2(2x-3) = 0$
$x = 0, \quad 2x - 3 = 0$
$x = \frac{3}{2}$

critical points are at
$f(0) = 0 \Rightarrow (0,0)$
$f(\frac{3}{2}) = (\frac{3}{2})^4 - 2(\frac{3}{2})^3$
$= -\frac{27}{16} \Rightarrow (\frac{3}{2}, -\frac{27}{16})$

4) intervals where f(x) is increasing ($f'(x) > 0$) and decreasing ($f'(x) < 0$); since critical points are at $x = 0$ and $x = \frac{3}{2}$, create the following intervals:

interval	$x < 0$	$0 < x < \frac{3}{2}$	$x > \frac{3}{2}$
test value	$x = -1$	$x = 1$	$x = 2$
$f'(x)$	$4(-1)^3 - 6(-1)^2 = -10 < 0$	$4(1)^3 - 6(1)^2 = -2 < 0$	$4(2)^3 - 6(2)^2 = 8 > 0$

$\therefore$ f(x) is decreasing on the interval $(-\infty, 3/2)$ and increasing on the interval $(3/2, \infty)$

5) inflection points at $f''(x) = 0 \Rightarrow f''(x) = 12x^2 - 12x$

continued →

$$12x^2 - 12x = 0$$
$$12x(x-1) = 0$$
$$x = 0, \quad x = 1$$

inflection points are at

$f(0) = 0 \Rightarrow (0,0)$

$f(1) = (1)^4 - 2(1)^3$
$= -1 \Rightarrow (1,-1)$

6) intervals where $f(x)$ is concave up ($f''(x) > 0$) and concave down ($f''(x) < 0$)

since inflection points are at $x = 0$ and $x = 1$, create the following intervals:

interval	$x < 0$	$0 < x < 1$	$x > 1$
test value	$x = -1$	$x = 0.5$	$x = 2$
$f''(x)$	$12(-1)^2 - 12(-1) = 24$ > 0	$12(0.5)^2 - 12(0.5) = -3$ < 0	$12(2)^2 - 12(2) = 24$ > 0

$\therefore$ $f(x)$ is concave up on the intervals $(-\infty, 0) \cup (1, \infty)$ and concave down on the interval $(0,1)$

7) end behaviour (limits at infinity)

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} x^4 - 2x^3 = \infty$ because of the term x^4

$\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} x^4 - 2x^3 = \infty$ for same reason.

we now have enough info to sketch the graph of $f(x)$

continued →

6. intercepts: $(0,0)$, $(2,0)$

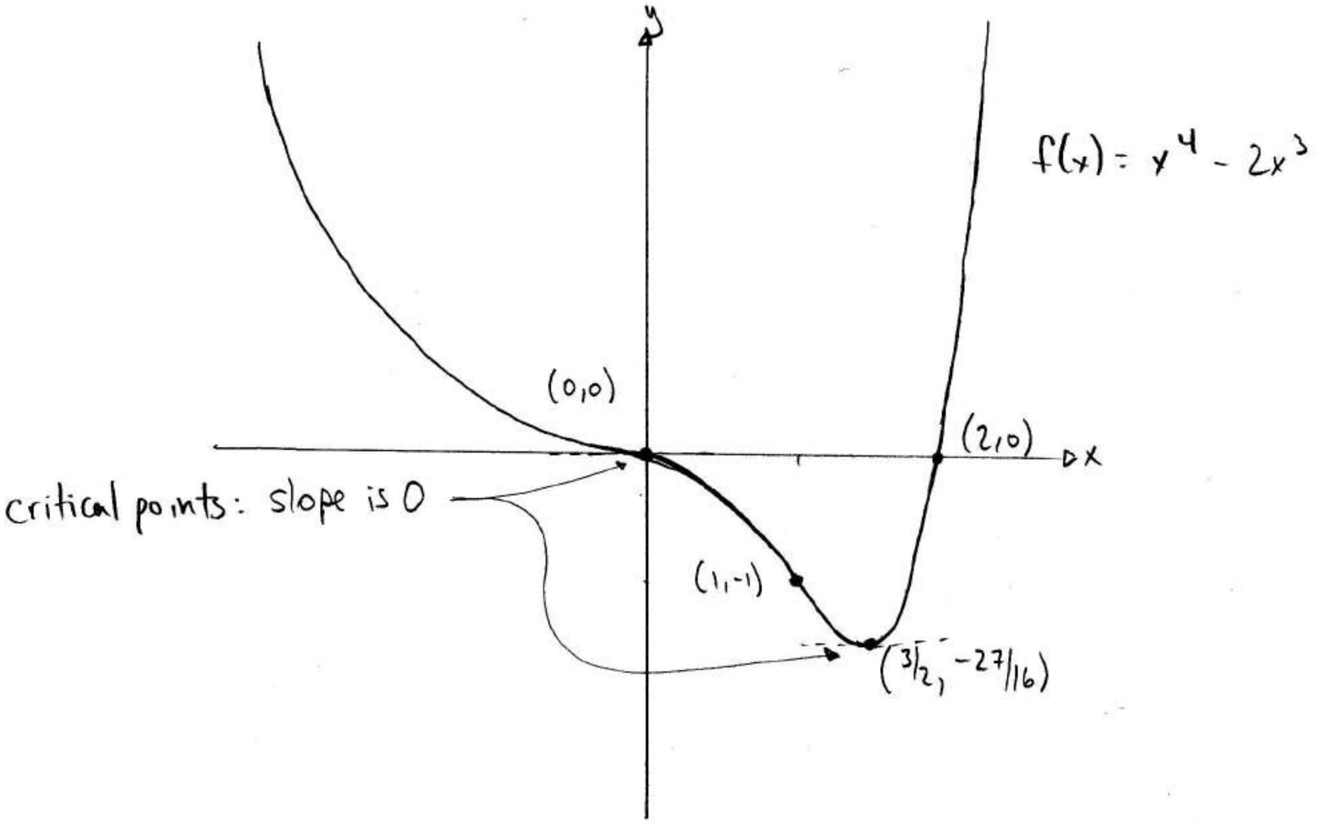
critical points: $(0,0)$, $(3/2, -27/16)$

inflection points: $(0,0)$, $(1,-1)$

$f(x)$ is: positive on $(-\infty,0),(2,\infty)$ negative on $(0,2)$

increasing on $(3/2,\infty)$, decreasing on $(-\infty, 3/2)$

concave up on $(-\infty,0)$, $(1,\infty)$, concave down on $(0,1)$



* Step 2 continued: find intervals where $f(x)$ is positive ($f(x) > 0$) and negative ($f(x) < 0$)

since x-intercepts are at $x = 0$ and $x = 2$, create the following intervals:

interval	$x < 0$	$0 < x < 2$	$x > 2$
test value	$x = -1$	$x = 1$	$x = 3$
$f(x)$	$(-1)^4 - 2(-1)^3 = 3$ > 0	$(1)^4 - 2(1)^3 = -1$ < 0	$(3)^4 - 2(3)^3 = 27$ > 0

$\therefore f(x)$ is positive on the intervals $(-\infty, 0) \cup (2,\infty)$, and negative on $(0,2)$