The University of British Columbia

Midterm Examination - November 9, 2011

MATH 104, Section 105

Closed book examination | Time: 50 minutes

Last Name ______________________________ **First** ________________________

Signature _________________

Student Number ___________________

Special Instructions:

No memory aids are allowed. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page. Where boxes are provided for answers, put your final answers in them.

Rules governing examinations

• Each candidate must be prepared to produce, upon request, a UBCcard for identification.

• Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

• No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.

• Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.

• Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

• Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		7
2		8
3		7
4		6
5		10
Total		38

[7] **1**. Consider the curve given by

$$y^2 + 2xy + x^3 = 4$$

(a) Evaluate $\frac{dy}{dx}$ at the points $(x, y) = (1, 1)$ and $(1, -3)$.

Use implicit differentiation to evaluate dy/dx

$$\begin{aligned} y^2 + 2xy + x^3 &= 4 \\ 2y\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y + 3x^2 &= 0 \\ (2y + 2x)\frac{dy}{dx} &= -2y - 3x^2 \\ \frac{dy}{dx} &= \frac{-2y - 3x^2}{2x + 2y} \end{aligned}$$

Substituting the points given

$$\boxed{\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} = \frac{-2(1) - 3(1)^2}{2(1) + 2(1)} = -\frac{5}{4}, \quad \left.\frac{dy}{dx}\right|_{(x,y)=(1,-3)} = \frac{-2(-3) - 3(1)^2}{2(1) + 2(-3)} = -\frac{3}{4}}$$

(b) Find the equations of the tangent lines to the curve at the points (x, y) given in a).

Using the answer from part a) for the slopes of the tangent lines, we just make the appropriate substitutions into the tangent line formula;

The tangent line at (1,1) is given by $y = -\frac{5}{4}(x - 1) + 1 \quad \rightarrow \quad \boxed{y = -\frac{5}{4}x + \frac{9}{4}}$

The tangent line at (1,-3) is given by $y = -\frac{3}{4}(x - 1) - 3 \quad \rightarrow \quad \boxed{y = -\frac{3}{4}x - \frac{9}{4}}$

[8] **2**. Air Canada sells tickets for flights between Montréal and Vancouver for \$500. Early market research has resulted in the following price-demand relationship

$$q = 10000e^{-kp}$$

where q is the daily demand, p is the price of a ticket and k is an unknown constant.

(a) If the daily demand at this price is 1000 customers, what is the value of the constant k. (Leave your answer in calculator-ready form).
Substitute $p = 500$, $q = 1000$ into the demand equation and solve for k.

$$\begin{aligned} 1000 &= 10000e^{-k(500)} \\ \frac{1}{10} &= e^{-500k} \\ \ln\left(\frac{1}{10}\right) &= -500k \\ -\ln(10) &= -500k \quad \rightarrow \quad \boxed{k = \frac{\ln(10)}{500}} \end{aligned}$$

(b) Find the elasticity of demand, $\epsilon(p) = \dfrac{p}{q}\dfrac{dq}{dp}$.

$$\begin{aligned} \epsilon(p) &= \frac{p}{q}\frac{dq}{dp} \\ &= \frac{p}{10000e^{-kp}}\frac{d}{dp}\left(10000e^{-kp}\right) \\ &= \frac{p}{10000e^{-kp}}10000e^{-kp}\cdot(-k) \\ &= -kp \quad \rightarrow \quad \boxed{\epsilon(p) = -kp} \end{aligned}$$

(c) If Air Canada raises their prices by 1%, what is the percentage change in demand? (Hint: Use elasticity of demand to answer this question).

$$\begin{aligned} \epsilon(p) &= \frac{\%\text{ change in demand}}{\%\text{ change in price}} \text{ at price } p. \\ \%\text{ change in demand} &= (\%\text{ change in price}) \times \epsilon(500) \\ &= 0.01 \times (-k \cdot 500) \\ &= 0.01 \times -\ln(10) \quad \rightarrow \quad \boxed{\%\text{ change in demand} = -\ln(10)\%} \end{aligned}$$

Therefore, demand will drop by $\ln(10)\%$ if the price is raised by 1% from \$500

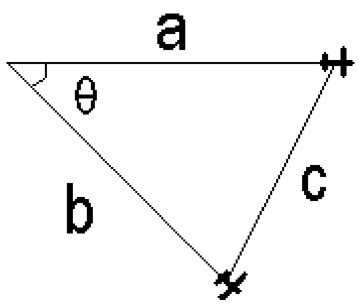
(d) Should Air Canada raise the price of their tickets to raise revenue?

$$|\epsilon(500)| = \ln(10) > 1$$

Therefore, demand is elastic and thus Air Canada should NOT raise the price to increase revenues. They should decrease the price.

[7] **3**. Two airplanes take-off from an airport at the same time. One of them is heading east and the other is heading south-east. Both are flying with speed 100 km/h. At what rate is the distance between them changing 1 hour after takeoff?

Hint: The cosine law says that given a triangle, the side lengths a, b, c satisfy $c^2 = a^2 + b^2 - 2ab\cos(\theta)$, where θ is the angle between side a and side b.)



We recognize from the wording of the problem and the given diagram that we want to evaluate dc/dt at $t = 1$ h. At $t = 1$ h, both planes have travelled 100 km. Thus, we have the following information.

$$\frac{da}{dt} = 100, \quad \frac{db}{dt} = 100, \quad a = 100, \quad b = 100, \quad \theta = \frac{\pi}{4}\,(= 45^\circ), \quad \frac{dc}{dt} = ?$$

We can even make the problem a little bit easier. If we recognize that since both planes take off from the same place at the same time and travel at the same speed, then we can set $a = b$.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos(\theta) \\ &= 2a^2 - 2a^2\cos(\theta) = 2a^2(1 - \cos(\theta)) \end{aligned}$$

Use the fact that $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

$$c^2 = 2a^2\left(1 - \frac{1}{\sqrt{2}}\right) = a^2(2 - \sqrt{2}) \tag{1}$$

Take the derivative of the equation *with respect to* t:

$$2c\frac{dc}{dt} = (2 - \sqrt{2})2a\frac{da}{dt} \quad \rightarrow \quad \frac{dc}{dt} = (2 - \sqrt{2})\frac{a}{c}\frac{da}{dt}$$

Now we see that we need the value of c at $t = 1$ h to finish evaluating dc/dt. Use Eqn (1)

$$c^2 = 100^2(2 - \sqrt{2}) \quad \rightarrow \quad c = 100\sqrt{2 - \sqrt{2}}$$

Plugging in all the appropriate constants in our equation for dc/dt gives

$$\boxed{\frac{dc}{dt} = (2 - \sqrt{2})\frac{(100)}{(100\sqrt{2 - \sqrt{2}})}(100) = 100\sqrt{2 - \sqrt{2}}}$$

Therefore, the distance between the planes is increasing at $100\sqrt{2 - \sqrt{2}}$ km/h when they are 1 h into their respective flights.

[6] **4**. You invest \$1,000 dollars in a savings account. Your banker tells you it will be worth \$10,000 in 30 years.

(a) Use the formula for continuously compounded interest to estimate the annual interest rate.

$$\begin{aligned} A &= Pe^{rt} \\ 10000 &= 1000e^{r(30)} \\ 10 &= e^{30r} \\ \ln(10) &= 30r \rightarrow \boxed{r = \frac{\ln(10)}{30}} \end{aligned}$$

(b) How long will you have to wait for your investment to be worth \$100,000?

$$\begin{aligned} 100000 &= 1000e^{rt} \\ 100 &= e^{rt} \\ \ln(100) &= rt \\ \ln(100) &= \frac{\ln(10)}{30}t \quad \rightarrow \quad \boxed{t = 30\frac{\ln(100)}{\ln(10)} = 30\log_{10}(100) = 60 \text{ years}} \end{aligned}$$

(c) Write an expression for the value of the investment in 15 years.

$$A = Pe^{rt} = 1000e^{\frac{\ln(10)}{30}15} = 1000e^{\frac{1}{2}\ln(10)} = 1000e^{\ln(10^{\frac{1}{2}})} = 1000(10^{1/2})$$

[10] **5**. Consider the function

$$f(x) = \frac{2x^2}{(x-1)^2}$$

(a) Find the vertical and horizontal asymptotes of $f(x)$, if any.

There is a vertical asymptote at $x = 1$ because $\lim_{x\to 1^-} f(x) = +\infty$, $\lim_{x\to 1^+} f(x) = +\infty$

There is a horizontal asymptote at $y = 2$ because $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2}{(x^2 - 2x + 1)}$

$$= \lim_{x\to\infty} \frac{2}{1 - 2/x + 1/x^2} = 2$$

(b) What is/are the critical point(s) of $f(x)$? Hint:

$$f'(x) = \frac{-4x}{(x-1)^3}$$

Critical points of f are values of x such that (1) $f'(x) = 0$ OR (2) $f'(x)$ DNE and $f(x)$ is defined. There are no points satisfying the second condition.

$$f'(x) = \frac{-4x}{(x-1)^3} = 0 \quad \rightarrow \quad \boxed{x = 0 \text{ is a critical point.}}$$

(c) For what values of x is $f(x)$ increasing? decreasing? Classify the critical point(s) from (b) as local maxima or local minima.

For this we need to check all intervals on the domain between the critical points and vertical asymptote points. This gives us three intervals to check: $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$.

$$\begin{aligned} \text{On } (-\infty, 0), f'(x) < 0, &\quad \rightarrow \quad f \text{ is decreasing on } (-\infty, 0) \\ \text{On } (0, 1), f'(x) > 0, &\quad \rightarrow \quad f \text{ is increasing on } (0, 1) \\ \text{On } (1, \infty), f'(x) < 0, &\quad \rightarrow \quad f \text{ is decreasing on } (1, \infty) \end{aligned}$$

Since f is decreasing left of $x = 0$ and increasing right of $x = 0$, the critical point at $x = 0$ must be a local minimum (by the first derivative test). Evaluating the corresponding y-value gives the minimum point: (0,0).

(d) Determine the inflection points of $f(x)$. For what values of x is $f(x)$ concave up? concave down? Hint:

$$f''(x) = \frac{4(2x+1)}{(x-1)^4}$$

Look for points where the $f''(x) = 0$ to determine possible inflection points (There are no points where f'' DNE and f is simultaneously defined).

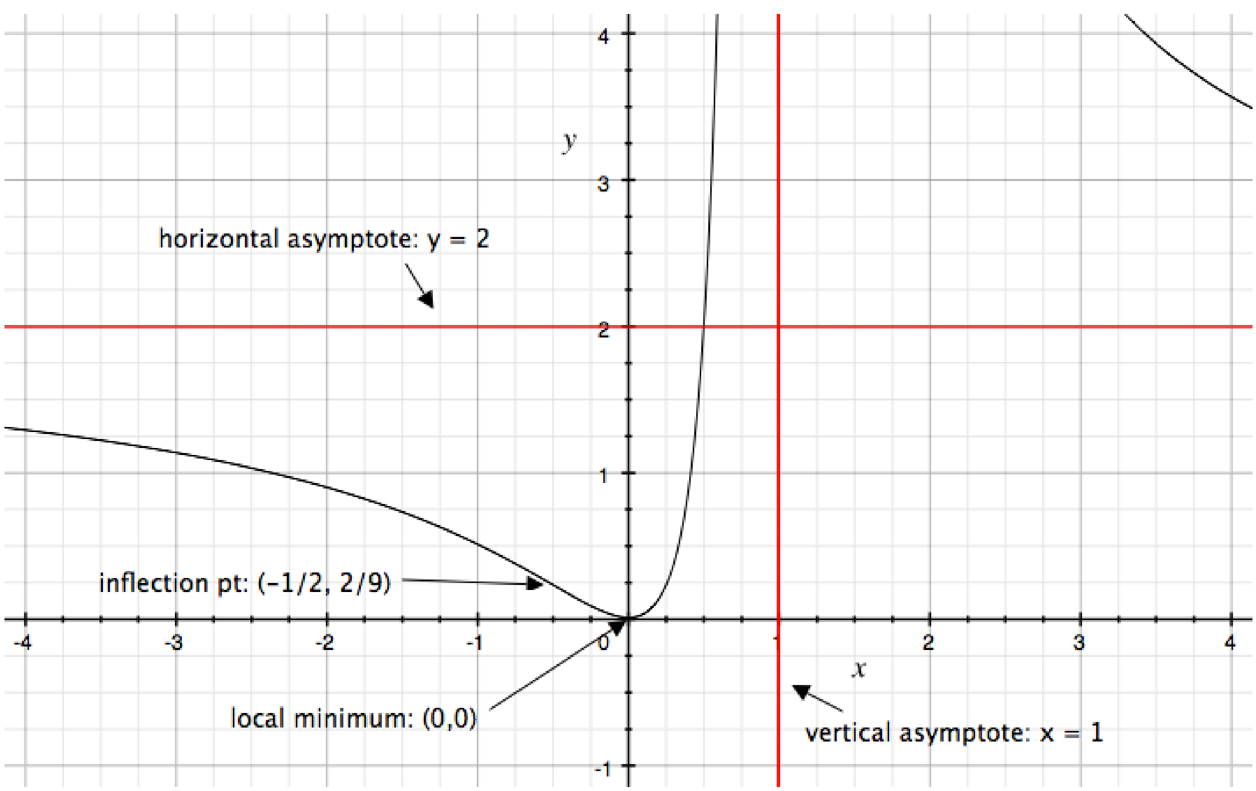
$$f''(x) = 0 \quad \rightarrow \quad 2x + 1 = 0 \quad \rightarrow \quad x = -\frac{1}{2}$$

As before we need to look at the sign of f'' around possible inflection points and asymptotes.

$$\begin{aligned} \text{On } (-\infty, -1/2), f''(x) < 0, &\quad \rightarrow \quad f \text{ is c.c down on } (-\infty, -1/2) \\ \text{On } (-1/2, 1), f''(x) > 0, &\quad \rightarrow \quad f \text{ is c.c up on } (-1/2, 1) \\ \text{On } (1, \infty), f''(x) > 0, &\quad \rightarrow \quad f \text{ is c.c up on } (1, \infty) \end{aligned}$$

Since the concavity changes sign at $x = -\frac{1}{2}$, we confirm that $x = -\frac{1}{2}$ is an inflection point. Evaluating f at the inflection point gives the coordinates of the inflection point: $(-1/2, 2/9)$.

(e) Make a sketch of $y = f(x)$, labeling all extreme points, inflection points and asymptotes.



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