MATH 209/4 all sections except EC: - Fundamental Mathematics II

Midterm - March 5, 2017 (1h30min)
Only approved calculators are permitted.

MARKS

[7] 1. (a) Find $\lim_{x\to 0} \frac{x}{\sqrt{x+49}-7}$.

[7] (b) Give functions $f(x)$ and $g(x)$ with the following properties:

(i) $\lim_{x\to 5} f(x) = 0$ (ii) $\lim_{x\to 5} g(x) = 0$ (iii) $\lim_{x\to 5} \frac{f(x)}{g(x)} = \frac{1}{5}$

[7] 2. Let $h(x) = 6 - x^3$. Work out the following in detail:

$$\lim_{s\to 0} \frac{[h(x+s) - h(x)]}{s}$$

[12] 3. (a) If $f(x) = -3x^{27} - 35$, find $f'(x)$. Do not simplify.

(b) If $g(x) = (5x^5 - 4)[ln(x^3) + 2]$, find $g'(x)$. Do not simplify.

(c) Find $h'(x)$ if $h(x) = \frac{e^5 - e^x}{[x^2 + \ln(x)]}$. Do not simplify.

(d) Find the value of dy if $y = x^4 + 2$, $x = 3$ and the change in x is 0.2.

[7] 4. A stock grows from ten dollars to twenty dollars in nine years. Find the associated annual rate of growth assuming that it is compounded continuously.

[7] 5. Suppose that a cost function is given by $C(x) = 3,000 + 7x$.

A student produced the following argument for finding the marginal average cost function: $C'(x) = 7$, $\frac{C'(x)}{x} = \frac{7}{x}$. Why is this answer incorrect?

[10] 6. A point is moving on the graph of $y^3 = x^2$. When the point is at $(-8, 4)$, its y-coordinate is increasing by 3 units per second. How fast is the x-coordinate changing at that moment?

[13] 7. Find the equation(s) of the tangent line(s) to the graph of $y - xy^2 + x^2 + 1 = 0$ at the point(s) where $x = 1$.

1a) $\lim_{x\to 0}\left(\frac{x}{\sqrt{x+49}-7}\right)\cdot\left(\frac{\sqrt{x+49}+7}{\sqrt{x+49}+7}\right) = \lim_{x\to 0}\frac{\cancel{x}(\sqrt{x+49}+7)}{\cancel{x}+\cancel{49}-\cancel{49}}$ rationalize

$\lim_{x\to 0} = \sqrt{x+49}+7 = \sqrt{0+49}+7 = \boxed{14}$

b) i) $\lim_{x\to 5} f(x) = 0$ ii) $\lim_{x\to 5} g(x) = 0$ iii) $\lim_{x\to 5}\frac{f(x)}{g(x)} = \frac{1}{5}$

$f(x) = 0$ or $f(x) = x-5$

or $f(x) = x^2-25$ or $f(x) = (x-5)^{10}$

or $f(x) = (x^2-25)^5$ or $f(x) = 3(x-5)$

or $f(x) = 10(x^2-25)$

Let $f(x) = x-5$ AND $g(x) = 5(x-5)$

i) $\lim_{x\to 5} f(x) = 5-5 = 0$

ii) $\lim_{x\to 5} g(x) = 5(5-5) = 0$

iii) $\lim_{x\to 5}\frac{f(x)}{g(x)} = \frac{1(\cancel{x-5})}{5(\cancel{x-5})} = \lim_{x\to 5}\frac{1}{5} = \frac{1}{5}$

2) let $h(x) = 6-x^3$ So $\boxed{h'(x) = \lim_{h\to 0}\frac{h(x+h)-h(x)}{h}}$ memorize!

$h(x+h) = 6-(x+h)^3 = 6-(x+h)(x+h)(x+h) \longrightarrow 6-(x+h)(x^2+2xh+h^2)$ $(x+h)^2$

$h(x) = 6-x^3$

$= 6-(x^3+2x^2h+xh^2+x^2h+2xh^2+h^3)$

$= 6-(x^3+3x^2h+3xh^2+h^3)$

$\lim_{h\to 0} = \frac{\cancel{6}-\cancel{x^3}-3x^2h-3xh^2-h^3-(\cancel{6}-\cancel{x^3})}{h}$

$\lim_{h\to 0} = \frac{\cancel{h}(-3x^2-3xh-h^2)}{\cancel{h}}$

$\lim_{h\to 0} = -3x^2-3x(0)-(0)^2$

$= -3x^2$

3) a) $f(x) = -3x^{27} - 35$

$f'(x) = -81x^{26} - 0 = -81x^{26}$

c) $\dfrac{x^5 + e^x}{e^{5x} + \ln x}$ use quotient rule!

$u = x^5 + e^x$ $\quad v = e^{5x} + \ln x$

$u' = 5x^4 + e^x$ $\quad v' = e^{5x}(5) + \frac{1}{x}$

$h'(x) = \dfrac{u'v - uv'}{v^2}$

$h'(x) = \dfrac{(5x^4 + e^x) - \frac{5e^{5x} +}{x}}{(e^{5x} + \ln x)^2}$

b) $g(x) = (6x^3 - 4) \cdot (\ln(x^2) + 2)$ use Product rule!

$u = 6x^3 - 4$ $\quad v = \ln x^2 + 2$

$u' = 18x^2$ $\quad v = 2\ln x + 2$ (ln rules)

$v' = \frac{2}{x}$

$g(x) = u'v + uv'$: Prod rule

$= 18x^2(\ln x^2 + 2) + (6x^3 - 4)\left(\frac{2}{x}\right)$

d) $y = x^4 + 2$ $\quad x = 3$ $\quad \Delta x = 0.2$

$dy = y'\Delta x$ memorize

$dy = y'\Delta x = (4x^3)\Delta x$

$dy = (4(3)^3)(0.2) = 108(0.2) = 21.6$

if they as to find Δy use

$\rightarrow \Delta y = y(x + \Delta x) - y(x) \approx dy$

$y(3 + 0.2) - y(3)$

$y(3.2) - y(3) = (3.2)^4 + 2 - (3^4 + 2) = 23.86$

extra Q's:

e) $f(x) = (8^x + \ln\pi - x^e)^5 \left(4\log_7 x + \dfrac{3}{2\sqrt[3]{x^5}}\right)$

$u = (8^x + \ln\pi - x^e)^5$ $\quad v = 4\log_7 x + \frac{3}{2}x^{-5/3}$

$u' = 5(8^x + \ln\pi - x^e)^4 \cdot (8^x \cdot \ln 8 + 0 - ex^{e-1})$ $\quad v' = \dfrac{4}{x\ln 7} - \dfrac{5}{2}x^{-8/3}$

constant $\quad$ memorize $\quad$ memorize

$f(x) = 5(8^x + \ln\pi - x^e)^4 \cdot (8^x \cdot \ln 8 - ex^{e-1})$ ......

4) $A = Pe^{rt}$

$20 = 10e^{9r}$

$2 = e^{9r}$

$\ln 2 = 9r$

$r = \dfrac{\ln 2}{9} = 0.07701 \times 100 = 7.70\%$

5) Correct Answer: $\bar{C} = \dfrac{8000 + 7x}{x} = 8000x^{-1} + 7$

$\bar{C}' = -8000x^{-2} = \dfrac{-8000}{x^2}$

His answer is incorrect because he found the average marginal cost and not the marginal average cost. He messed up the order of the calculations

6) equation → plugin → derive implicitly → plug in.

Step1: $y^3 = x^2$ Step2: $(-8,4)$ Step3: $3y^2y' = 2xx'$ Step4: $3(4)^2(3) = 2(-8)x$ → Pos(+) cuz inc?

$x' = -9$ units/sec

?

7)

Slope = derivative

$y - xy^2 + x^2 + 1 = 0$

$u = x$ $v = y^2$

$u' = 1$ $v' = 2yy'$

$1y' - (y^2 + 2xyy') + 2x = 0$

$y' - 2xyy' = y^2 - 2x$

$y'(1 - 2xy) = y^2 - 2x$

$$y' = \frac{y^2 - 2x}{1 - 2xy}$$

POINT: $x \to f(x)$, $x = 1$ → $y - y^2 + 1 + 1 = 0$

$0 = y^2 - y - 2$

$0 = (y-2)(y+1)$

$y = 2, y = -1$

↳ two tangent lines because two points

2 Points! $(1,2)$ & $(1,-1)$

$$\text{SLOPE} = f'(x) = \frac{y^2 - 2x}{1 - 2xy}$$

Slope one: $y'\Big|_{(1,2)} = \frac{2^2 - 2(1)}{1 - 2(1)(2)} = -\frac{2}{3}$

Tangent one:-

$y = -\frac{2}{3}(x-1) + 2$

$y = m(x - x_1) + y_1$ → plug in blanks

Slope two:- $y'\Big| = \frac{(-1)^2 - 2(1)}{1 - 2(1)(-1)} = -\frac{1}{3}$

Tangent two:-

$y = -\frac{1}{3}(x-1) - 1$