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STUDENT NUMBER: ____________________

University of Toronto
Faculty of Arts and Sciences

April 2019 Examinations

MAT223 Practice Final

Duration: 3 hours
Aids Allowed: None

Exam Reminders:

- Fill out your name, student number, and email address at the top of this page.
- Do not begin writing the actual exam until the announcements have ended and the Exam Facilitator has started the exam.
- As a student, you help create a fair and inclusive writing environment. If you possess an unauthorized aid during an exam, you may be charged with an academic offence.
- Turn off and place all cell phones, smart watches, electronic devices, and unauthorized study materials in your bag under your desk. If it is left in your pocket, it may be an academic offence.
- When you are done with your exam, raise your hand for someone to come and collect your exam. Do not collect your bag and jacket before your exam is handed in.
- If you are feeling ill and unable to finish your exam, please bring it to the attention of an Exam Facilitator so it can be recorded before leaving the exam hall.
- In the event of a fire alarm, do not check your cell phone when escorted outside.

Special Instructions:

- Write legibly and darkly.
- Cross out any work you do not wish to have scored, and clearly indicate if there is work on another page you want scored.
- Show all of your work. Unsupported answers may not earn credit.

Exam Format and Grading Scheme:

Answers must be written on the examination paper.

Question:	1	2	3	4	5	6	7	8	9	10	11	12	Total
Points:	28	8	8	16	8	5	6	13	14	6	10	15	137

Students must hand in all examination materials at the end

1. Complete the following sentences with a mathematically correct definition. No marks will be awarded for a "close" but incorrect definition.

   (a) (2 points) $M \subseteq \mathbb{R}^n$ is a *subspace* if

   (b) (2 points) The vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ are *linearly independent* if

   (c) (2 points) The vectors $\vec{x}$ and $\vec{y}$ are *orthogonal* if

   (d) (2 points) The *rank* of a linear transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^m$ is

   (e) (2 points) The *range* of a linear transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^m$ is

   (f) (2 points) A linear transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^m$ is *one-to-one* if

   (g) (2 points) Let $\mathcal{B}$ be a basis for $\mathbb{R}^n$ and let $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$. The matrix $[\mathcal{T}]_{\mathcal{B}}$ is

(h) (2 points) E is a 3×3 *elementary matrix* if

(i) (2 points) A matrix A is *diagonalizable* if

(j) (2 points) A matrix A is invertible if

(k) (2 points) The matrices A and B are *similar* if

(l) (2 points) The *eigenspace*, E, of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with corresponding eigenvalue λ is

(m) (2 points) The *characteristic polynomial* of the matrix A is

(n) (2 points) The *determinant* of a linear transformation $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ is

2. Let S be the set of all solutions to the system $\begin{cases} x_1 + 6x_2 + 2x_3 - 5x_4 - 2x_5 = -4 \\ \qquad\qquad 2x_3 - 8x_4 - 1x_5 = 3 \\ \qquad\qquad\qquad\qquad x_5 = 7. \end{cases}$

(a) (3 points) Express S in vector form.

(b) (3 points) Is S a subspace? Prove your answer.

(c) (2 points) Is S a *translated* subspace? If so, find a subspace V and a vector $\vec{w}$ so that $S = V + \{\vec{w}\}$.

3. Let $\mathcal{E}$ be the standard basis for $\mathbb{R}^4$. You know the following about the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$.

$$T \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} w - x \\ x - y \\ y - z \\ z - w \end{bmatrix}_{\mathcal{E}}$$

(a) (2 points) Find a matrix for T in the standard basis (i.e., find $[T]_{\mathcal{E}}$).

(b) (2 points) Give a basis for the null space of T.

(c) (2 points) Give a basis for the range of T

(d) (2 points) What is the volume of the image of the unit 4-cube under the transformation T?

4. For each of the following, give an example if possible. Otherwise, explain why it is impossible.

(a) (2 points) A *non-diagonal* 2×2 matrix A with eigenvalues 7 and 101.

(b) (2 points) A 6×4 matrix B whose column space is three dimensional.

(c) (2 points) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ whose null space is the y-axis.

(d) (2 points) A subspace of dimension 3 that is the span of 3 linearly independent vectors in $\mathbb{R}^5$.

(e) (2 points) A subspace of dimension 2 that is the span of 3 linearly independent vectors in $\mathbb{R}^3$.

(f) (2 points) A linear transformation $X : \mathbb{R}^5 \to \mathbb{R}$ whose nullity is 5.

(g) (2 points) A 3×2 matrix A with positive determinant.

(h) (2 points) A 3×3 matrix A with 5 distinct eigenvectors (not necessarily linearly independent).

5. Let $P \subset \mathbb{R}^4$ be the subspace with equation $2x + 3y - z + w = 0$, let $Q \subset \mathbb{R}^4$ be the hyper-plane with equation $x + y - z = 0$, let $R \subset \mathbb{R}^4$ be the hyper-plane with equation $x - y + w = 0$, and let $l = P \cap Q \cap R$

   (a) (2 points) Is l a subspace? If so, what is the dimension? Explain.

   (b) (2 points) Give an example of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ so that $l = null(T)$.

   (c) (2 points) Give an example of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ so that $l = range(T)$.

   (d) (2 points) Find an eigenvalue and a corresponding eigenvector of the linear transformation above.

6. Suppose we have a 10×10 matrix A such that the matrix-vector equation $A\vec{x} = \vec{b}$ is always consistent.

(a) (1 point) Does $A\vec{x} = \vec{b}$ always have a unique solution? Justify your answer.

(b) (2 points) What are the possible dimensions of the row space of matrix A? Justify your answer.

(c) (2 points) What are the possible dimensions of the null space of matrix A? Justify your answer.

7. Let M be a 3×3 matrix with three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

   (a) (3 points) Prove that $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent.

   (b) (3 points) Is M diagonalizable? Justify your answer.

8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation . Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Let $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ where $c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{E}}$, $c_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}_{\mathcal{E}}$, $c_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{E}}$, and suppose $[T]_{\mathcal{E}} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

(a) (3 points) What are the eigenvalues of $[T]_{\mathcal{E}}$?

(b) (2 points) Find the matrix Q which changes vectors from the $\mathcal{C}$ basis to the $\mathcal{E}$ basis.

(c) (2 points) Find a matrix that changes vectors from the $\mathcal{E}$ basis to the $\mathcal{C}$ basis.

(d) (2 points) Find a matrix D such that $[T]_{\mathcal{E}} = QDQ^{-1}$.

(e) (2 points) Calculate D^{10} (where D is the matrix you found in the previous part).

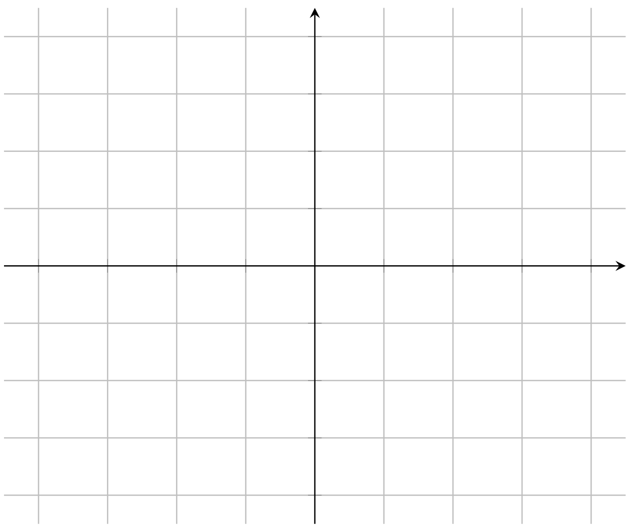
(f) (2 points) Is $[T]_{\mathcal{E}}^{10}$ diagonalizable? If so, find a matrix P and a diagonal matrix A so that $[T]_{\mathcal{E}}^{10} = PAP^{-1}$.

9. Let $S \subseteq \mathbb{R}^2$ be the sides of the triangle with corners $\{\vec{0}, \vec{e}_1, \vec{e}_2\}$ and C be the unit circle centered at the origin, let T be the linear transformation with standard matrix

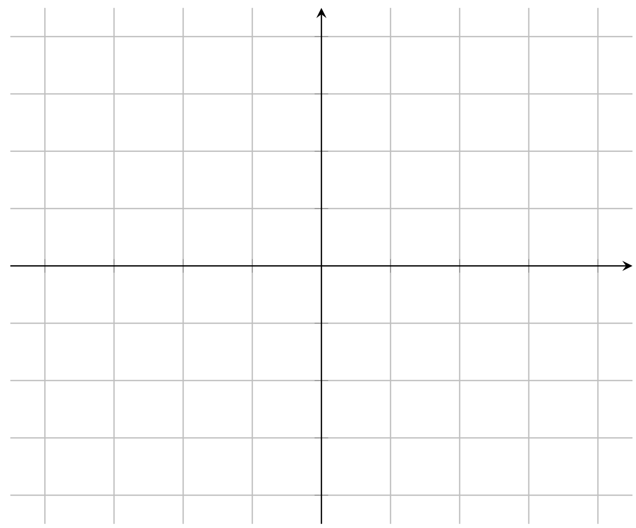
$$M = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$$

Draw the following subsets of $\mathbb{R}^2$.

(a) (2 points) $T(S) \cup T(C)$



(b) (2 points) $null(T)$



(c) (2 points) Find the area of $T(S)$ and $T(C)$

(d) (4 points) Describe, in complete sentences, what T does geometrically.

(e) (4 points) If possible, express T as a composition of two, simpler linear transformations.

10. (6 points) Ron and Jake are arguing about the following claim by a student they met at a food truck:

> *Any square matrix can be written as a product* $E_k \cdots E_1 P$ *where each* E_i *is an elementary matrix and* P *is the matrix for a projection.*

Ron thinks the claim is true because elementary matrices can be used as a replacement for row reduction. Jake is not yet convinced.

Explain to Ron and Jake, using complete English sentences, whether or not the claim is true by either filling out the details of Ron's argument, or showing that the claim is false and explaining where Ron's argument went wrong.

11. In this question you will be working with a new definition.
A square matrix is *positive definite* if $(A\vec{x}) \cdot \vec{x} > 0$ for all $\vec{x} \neq \vec{0}$.

(a) (2 points) Given an example of such a matrix.

(b) (3 points) What is the null space of a positive definite matrix? Justify your answer.

(c) (5 points) If a 2×2 matrix has two distinct positive eigenvalues, could the matrix positive definite? Justify your answer.

12. Suppose a matrix A is such that

(i) the matrix-vector equation $A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ always has a unique solution; and

(ii) $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) (1 point) What is the size of the matrix A.

(b) (2 points) What is the dimension of the row space of matrix A? Justify your answer.

(c) (3 points) Is it possible that both $\vec{e}_1$ and $\vec{e}_2$ are in the range of T_A (the transformation induced by A)?

Consider property (iii)

(iii) $A \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

(d) (3 points) How many matrices exist satisfying properties (i), (ii), and (iii)? Justify your answer.

(e) (3 points) How many matrices exist satisfying properties (i) and (ii)? Justify your answer.

(f) (3 points) How many matrices exist satisfying properties (ii) and (iii)? Justify your answer.

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If you would like work on this page scored, then clearly indicate to which question the work belongs and indicate on the page containing the original question that there is work on this page to score.

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