

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	Number	Section(s)
Mathematics	204	All

Examination	Date	Pages
Final	December 2017	2

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Special Instructions

- ▷ Only approved calculators are allowed.
 - ▷ Justify all your answers.
 - ▷ Answer all questions. All questions have equal value.
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1. Solve the system by Gauss-Jordan elimination

$$\begin{aligned}x - 2y + z &= 2 \\2x - 4y + 2z &= 4 \\5x - y + 2z &= 13.\end{aligned}$$

2. Find the inverse of the matrix $A = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 3 & 0 \\ 4 & 0 & -6 \end{pmatrix}$

and solve for X if $AX = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 1 & 2 & 4 & 5 \end{pmatrix}$.

3. Find the matrix of cofactors of $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 7 & 8 \\ 4 & -1 & 4 \end{pmatrix}$

and find the value of $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{13}$.

4. (a) Find the determinant of $A = \begin{pmatrix} 6 & -1 & 0 & 4 \\ 3 & 3 & -2 & 0 \\ 0 & 1 & 8 & 6 \\ 2 & 3 & 0 & 4 \end{pmatrix}$

(b) Solve for x if $\begin{vmatrix} x & 1 & -2 \\ 1 & -1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 7$.

5. (a) Find an equation of the plane containing the point $P(2, 1, -1)$ and the line $x = 3t - 2, y = -t + 4, 7z = 2t + 1$.
- (b) Find the point of intersection of the line $x = 2t - 1, y = 3t - 2, z = t + 4$ and the plane $x + 2y + 3z = -4$.
6. Show that the points $P(1, 2, 4), Q(-1, 0, 3), R(-2, -4, 1), S(-3, -2, 2)$ lie in the same plane.
7. Find the standard matrix for the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given that $T(1, 2) = (1, 6), T(2, 1) = (1, 4)$.
8. Given the vectors $x_1 = (1, 2, 3, 4), x_2 = (1, 0, 1, 4), x_3 = (2, 1, 0, 3), x_4 = (1, 1, 2, 6), x_5 = (-1, 2, -3, 4)$, find a basis of the subspace of \mathbb{R}^4 spanned by x_1, x_2, x_3, x_4, x_5 .
9. In question 8, find coefficients a, b, c, d, e (not all zero) such that $ax_1 + bx_2 + cx_3 + dx_4 + ex_5 = 0$.
10. Let $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

MATH. 204.

Final Exam Solutions.

Dec. 2017.

$$1. \begin{bmatrix} 1 & -2 & 1 & | & 2 \\ 2 & -4 & 2 & | & 4 \\ 5 & -1 & 2 & | & 13 \end{bmatrix} \xrightarrow{\substack{-2R_1 + R_2 \rightarrow R_2 \\ -5R_1 + R_3 \rightarrow R_3}} \begin{bmatrix} 1 & -2 & 1 & | & 2 \\ 0 & 0 & 0 & | & 0 \\ 0 & 9 & -3 & | & 3 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{9}R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & 1 & | & 2 \\ 0 & 1 & -1/3 & | & 1/3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \xrightarrow{2R_2 + R_1 \rightarrow R_1}$$

$$\begin{bmatrix} 1 & 0 & 1/3 & | & 8/3 \\ 0 & 1 & -1/3 & | & 1/3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{aligned} x &= \frac{8}{3} - \frac{1}{3}z \\ y &= \frac{1}{3} + \frac{1}{3}z \end{aligned}$$

Let $z = t$.

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{8}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} t.$$

2.

$$A^{-1} = \begin{bmatrix} -1 & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{4}{9} & -\frac{1}{9} \\ -\frac{2}{3} & -\frac{2}{9} & \frac{1}{18} \end{bmatrix}.$$

$$X = \begin{bmatrix} -\frac{4}{3} & \frac{1}{3} & 0 & -\frac{4}{3} \\ \frac{10}{9} & \frac{2}{9} & \frac{1}{3} & \frac{13}{9} \\ -\frac{19}{8} & -\frac{1}{9} & -\frac{2}{3} & -\frac{31}{18} \end{bmatrix}.$$

3. Cofactor Matrix:

$$\begin{bmatrix} 36 & 20 & -31 \\ -2 & -4 & 1 \\ -14 & -2 & 7 \end{bmatrix}.$$

\therefore) $(1)(-2) + (0)(-4) + (2)(-31) = -64.$

4. A.

$$\det(A) = 560.$$

$$B. \begin{bmatrix} x & 1 & -2 \\ 1 & -1 & 1 \\ -1 & 0 & 2 \end{bmatrix} = 7$$

$$[-2x - 1 + 0] - [-2 + 0 + 2] = 7$$

$$-2x - 1 = 7$$

$$x = -4.$$

$$5.A. P(2, 1, -1).$$

$$\vec{a} = \left(3, -1, \frac{2}{7}\right).$$

$$Q = \left(-2, 4, \frac{1}{7}\right).$$

$$\vec{PQ} = \left(-4, 3, \frac{8}{7}\right).$$

$$\vec{n} = \vec{a} \times \vec{PQ} = \begin{vmatrix} 3 & -1 & \frac{2}{7} \\ -4 & 3 & \frac{8}{7} \end{vmatrix}$$

$$\vec{n} = \left(-2, -\frac{32}{7}, 5\right).$$

\therefore

$$-2(x-2) - \frac{32}{7}(y-1) + 5(z+1) = 0$$

$$-2x - \frac{32}{7}y + 5z = -\frac{95}{7}.$$

$$B. \quad x + 2y + 3z = -4$$

$$(2x-1) + 2(3x-2) + 3(x+4) = -4$$

$$11x = -11$$

$$x = -1 \Rightarrow x = -3$$

$$y = -5$$

$$z = 3.$$

$$6. \vec{PQ} = (-2, -2, -1),$$

$$\vec{PR} = (-3, -6, -3).$$

$$\vec{PS} = (-4, -4, -2).$$

$$\vec{PQ} \cdot (\vec{PR} \times \vec{PS})$$

$$\vec{PQ} \cdot (0, 6, -12)$$

$$(-2, -2, -1) \cdot (0, 6, -12) = 0.$$

$$7. \text{ Let } T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} \Rightarrow \begin{aligned} a + 2b &= 1 \\ c + 2d &= 6. \end{aligned}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \Rightarrow \begin{aligned} 2a + b &= 1 \\ 2c + d &= 4 \end{aligned}$$

$$\therefore a = \frac{1}{3} \quad c = \frac{2}{3}$$
$$b = \frac{1}{3} \quad d = \frac{8}{3}.$$

$$\therefore T = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & 8 \end{bmatrix}.$$

8 and 9.

$$\left[\begin{array}{ccccc} 1 & 1 & 2 & 1 & -1 \\ 2 & 0 & 1 & 1 & 2 \\ 3 & 1 & 0 & 2 & -3 \\ 4 & 4 & 3 & 6 & 4 \end{array} \right] \begin{array}{l} -2R_1 + R_2 \rightarrow R_2 \\ -3R_1 + R_3 \rightarrow R_3 \\ \hline -4R_1 + R_4 \rightarrow R_4 \end{array}$$

$$\left[\begin{array}{ccccc} 1 & 1 & 2 & 1 & -1 \\ 0 & -2 & -3 & -1 & 4 \\ 0 & -2 & -6 & -1 & 0 \\ 0 & 0 & -5 & 2 & 8 \end{array} \right] \begin{array}{l} -\frac{1}{2}R_2 \rightarrow R_2 \\ -R_2 + R_3 \rightarrow R_3 \\ \hline -R_2 + R_1 \rightarrow R_1 \end{array}$$

$$\left[\begin{array}{ccccc} 1 & 0 & 1/2 & 1/2 & 1 \\ 0 & 1 & 3/2 & 1/2 & -2 \\ 0 & 0 & -3 & 0 & -4 \\ 0 & 0 & -5 & 2 & 8 \end{array} \right] \begin{array}{l} -\frac{1}{3}R_3 \rightarrow R_3 \\ 5R_3 + R_4 \rightarrow R_4 \\ \hline -\frac{3}{2}R_3 + R_2 \rightarrow R_2 \\ -\frac{1}{2}R_3 + R_1 \rightarrow R_1 \end{array}$$

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & 1/2 & 1/3 \\ 0 & 1 & 0 & 1/2 & -4 \\ 0 & 0 & 1 & 0 & 4/3 \\ 0 & 0 & 0 & 2 & \frac{44}{3} \end{array} \right] \begin{array}{l} \frac{1}{2}R_4 \rightarrow R_4 \\ -\frac{1}{2}R_4 + R_1 \rightarrow R_1 \\ \hline -\frac{1}{2}R_4 + R_2 \rightarrow R_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -10/3 \\ 0 & 1 & 0 & 0 & -23/3 \\ 0 & 0 & 1 & 0 & 4/3 \\ 0 & 0 & 0 & 1 & \frac{22}{3} \end{bmatrix}.$$

Basis vectors: $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$.

$$\therefore a = -\frac{10}{3}$$

$$b = -\frac{23}{3}$$

$$c = \frac{4}{3}$$

$$d = \frac{22}{3}$$

$$e = -1.$$

10.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

$$\det(\lambda I - A) = 0$$

$$\det \left(\begin{bmatrix} (\lambda-1) & -3 & -3 \\ 3 & (\lambda+5) & 3 \\ -3 & -3 & (\lambda-1) \end{bmatrix} \right) = 0$$

$$[(\lambda-1)^2(\lambda+5) + 27 + 27] - [9(\lambda+5) - 9(\lambda-1) - 9(\lambda-1)] = 0$$

$$(\lambda-1)(\lambda+2)^2 = 0 \quad \text{C.E.}$$

$$\lambda = 1; -2.$$

$$\lambda = 1:$$

$$\begin{bmatrix} 0 & -3 & -3 \\ 3 & 6 & 3 \\ -3 & -3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

$$\text{E.V. for } \lambda = 1: \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

$$\lambda = -2: \begin{bmatrix} -3 & -3 & -3 \\ 3 & 3 & 3 \\ -3 & -3 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$\text{E.V. for } \lambda = -2: \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}; \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \Rightarrow P^{-1}AP = D.$$