University of Ottawa

MAT 1332, Practice midterm (longer than the actual midterm)

February 2019

These are questions to help you see the scope of the first midterm exam. The format of the midterm will be as follows:

- About 33% multiple-choice and short answer; 67% long answer with full justification required. About 27 points in total; worth 20% of your final grade.
- About 7 questions, some with multiple parts.
- Covering material from : integration: substitution, integration by parts, partial fractions, Riemann sums, improper integrals; differential equations: checking solutions, solving separable differential equations; autonomous differential equations: equilibria, phase line diagrams, stability; complex numbers; matrix algebra.
- Expect to be asked to interpret your mathematical analysis in the context of the application at hand.

QUESTION A1. Compute the following integrals:

(a) $\int \frac{x^{3/2} - 5x}{\sqrt{x}} dx$

$$\begin{aligned} \int \frac{x^{3/2} - 5x}{\sqrt{x}} dx &= \int x - 5\sqrt{x}\, dx \\ &= \frac{1}{2}x^2 - 5\left(\frac{1}{3/2}x^{3/2}\right) + c \\ &= \frac{1}{2}x^2 - \frac{10}{3}x^{3/2} + c \end{aligned}$$

for c an arbitrary constant.

(b) $\int \left(\sin x + 4x^2 - \frac{6x}{\sqrt[3]{x}}\right) dx$

$$\begin{aligned} \int \left(\sin x + 4x^2 - \frac{6x}{\sqrt[3]{x}}\right) dx &= \int \sin(x)\, dx + 4\int x^2\, dx - 6\int x^{2/3}\, dx \\ &= -\cos(x) + \frac{4}{3}x^3 - 6\left(\frac{1}{5/3}x^{5/3}\right) + c \\ &= -\cos(x) + \frac{4}{3}x^3 - \frac{18}{5}x^{5/3} + c \end{aligned}$$

for c an arbitrary constant.

(c) $\int \sec^2 x \tan x \, dx$

Use the substitution $u = \tan(x)$, $du = \sec^2(x)\, dx$. Then we have

$$\int \sec^2(x) \tan(x)\, dx = \int u \, du = \frac{1}{2}u^2 + c = \frac{1}{2}\tan^2(x) + c$$

for c an arbitrary constant.

(d) $\int (x^3 + x)^{10}(3x^2 + 1) dx$

To avoid multiplying out a horrible expression, we try the substitution $u = x^3 + x$, $du = (3x^2 + 1)\, dx$. This works well:

$$\int (x^3 + x)^{10}(3x^2 + 1)\, dx = \int u^{10}\, du = \frac{1}{11}u^{11} + c = \frac{1}{11}(x^3 + x)^{11} + c$$

for c an arbitrary constant.

(e) $\int x^4 \ln x dx$

This is a product of two distinct functions, with no obvious composition, so we try integration by parts.

Let $u = \ln(x)$ and $dv = x^4\, dx$. Then $du = \frac{1}{x}\, dx$ and $v = \frac{1}{5}x^5$. Then we have

$$\begin{aligned}
\int x^4 \ln x \, dx &= \frac{1}{5}x^5 \ln(x) - \int \frac{1}{x} \cdot \frac{1}{5}x^5 \, dx \\
&= \frac{1}{5}x^5 \ln(x) - \frac{1}{5}\int x^4 \, dx \\
&= \frac{1}{5}x^5 \ln(x) - \frac{1}{5}\left(\frac{1}{5}x^5\right) + c \\
&= \frac{1}{5}x^5 \ln(x) - \frac{1}{25}x^5 + c,
\end{aligned}$$

where c is an arbitrary constant.

(f) $\int x \sin(x/2) dx$

We could do the substitution $w = x/2$, giving $dw = \frac{1}{2}\, dx$. Then since $x = 2w$ and $dx = 2\, dw$ we have

$$\int x \sin(x/2)\, dx = \int (2w)\sin(w)\, 2\, dw = 4\int w \sin(w)\, dw.$$

Now we apply integration by parts, with $u = w$, $dv = \sin(w)\, dw$, so that $du = dw$ and $v = -\cos(w)$. This gives

$$\int w \sin(w)\, dw = -w\cos(w) - \int (-\cos(w))\, dw = -w\cos(w) + \int \cos(w)\, dw = -w\cos(w) + \sin(w) + c$$

for c an arbitrary constant. Finally, putting this together we get

$$\int x \sin(x/2)\, dx = 4 \int w \sin(w)\, dw = 4(-w \cos(w)+\sin(w)+c) = -4w\cos(x)+4\sin(w)+d$$

where d is some arbitrary constant (equal to $4c$, but c was arbitrary, too).

(g) $\displaystyle\int \frac{e^{1/x}}{5x^2} dx$

We use the substitution $u = 1/x = x^{-1}$, $du = -x^{-2}\, dx$. Then $(-1)\, du = \frac{1}{x^2}\, dx$ so

$$\begin{aligned}\int \frac{e^{1/x}}{5x^2} dx &= \int \frac{1}{5} e^u (-1)\, du \\ &= -\frac{1}{5} \int e^u\, du \\ &= -\frac{1}{5} e^u + c \\ &= -\frac{1}{5} e^{1/x} + c,\end{aligned}$$

where c is an arbitrary constant.

QUESTION A2. Use 5 equal subintervals and (a) left endpoints, (b) midpoints, (c) right endpoints to calculate Riemann sums of $f(x) = \sin(x)$ on $[0, \pi/2]$.

We are integrating over $[a, b] = [0, \pi/2]$. For 5 subintervals we need $\Delta x = \frac{b-a}{5} = \pi/10$. Therefore the endpoints of the subintervals are

$$x_0 = 0, x_1 = \frac{\pi}{10}, x_2 = \frac{2\pi}{10}, x_3 = \frac{3\pi}{10}, x_4 = \frac{4\pi}{10}, x_5 = \frac{5\pi}{10} = \frac{\pi}{2},$$

where we could simplify some of these fractions, of course.

(a) For the left endpoint rule we use

$$\begin{aligned}\sum_{i=0}^{4} f(x_i)\Delta x &= \sin(x_0)\Delta x + \sin(x_1)\Delta x + \sin(x_2)\Delta x + \sin(x_3)\Delta x + \sin(x_4)\Delta x \\ &= \Delta x(\sin(0) + \sin(\pi/10) + \sin(\pi/5) + \sin(3\pi/10) + \sin(2\pi/5)) \\ &= \frac{\pi}{10}(2.6569) \\ &= 0.8347.\end{aligned}$$

(c) For the right endpoint rule we use

$$\begin{aligned}\sum_{i=1}^{5} f(x_i)\Delta x &= \sin(x_1)\Delta x + \sin(x_2)\Delta x + \sin(x_3)\Delta x + \sin(x_4)\Delta x + \sin(x_5)\Delta x \\ &= \Delta x(\sin(\pi/10) + \sin(\pi/5) + \sin(3\pi/10) + \sin(2\pi/5) + \sin(\pi/2)) \\ &= \frac{\pi}{10}(3.6569) \\ &= 1.1488.\end{aligned}$$

(b) To apply the midpoint rule, we need to use the midpoints of each interval, which are given by $\frac{1}{2}(x_i + x_{i+1})$ in each case. This gives

$$\begin{aligned}\sum_{i=0}^{4} f(\frac{1}{2}(x_i + x_{i+1}))\Delta x &= \sin(\frac{1}{2}(x_0 + x_1))\Delta x + \sin(\frac{1}{2}(x_1 + x_2))\Delta x + \\ &\quad + \sin(\frac{1}{2}(x_2 + x_3))\Delta x + \sin(\frac{1}{2}(x_3 + x_4))\Delta x + \sin(\frac{1}{2}(x_4 + x_5))\Delta x \\ &= \Delta x(\sin(\pi/20) + \sin(3\pi/20) + \sin(5\pi/20) + \sin(7\pi/20) + \sin(9\pi/20)) \\ &= \frac{\pi}{10}(3.1962) \\ &= 1.0041.\end{aligned}$$

We can also integrate exactly in this case and deduce that the exact answer is 1.

Question A3. A population grows according to the differential equation

$$\frac{dN}{dt} = 2\frac{e^{-\sqrt{t}/2}}{\sqrt{t}}$$

.

Determine the net change of this population between $t = 4$ and $t = 16$.

A) $\frac{2}{3}(e^{-2} - e^{-20})$ B) $8(e^{-1} - e^{-2})$ C) $8(e^{-4} - e^{-16})$
D) $6(e^{-1} - e^{-2})$ E) $\frac{5}{3}(1 - e^{-15})$ F) 0

The net change from 4 to 16 is the definite integral $\int_4^{16} \frac{dN}{dt}\, dt$. We compute the indefinite integral first.

Use the substitution $u = -\frac{1}{2}\sqrt{t}$, so that $du = -\frac{1}{4}t^{-1/2}\, dt$. Then $\frac{1}{\sqrt{t}}\, dt = -4\, du$ so

$$\begin{aligned}\int 2\frac{e^{-\sqrt{t}/2}}{\sqrt{t}}\, dt &= 2\int e^u(-4)\, du \\ &= -8\int e^u\, du \\ &= -8e^u + c \\ &= -8e^{-\sqrt{t}/2} + c,\end{aligned}$$

for c an arbitrary constant.

Now evaluate the definite integral using the choice $-8e^{-\sqrt{t}/2}$ of antiderivative:

$$\begin{aligned}\int_4^{16} 2\frac{e^{-\sqrt{t}/2}}{\sqrt{t}}\, dt &= \left[-8e^{-\sqrt{t}/2}\right]_4^{16} \\ &= -8e^{-\sqrt{16}/2} - (-8)e^{-\sqrt{4}/2} \\ &= -8e^{-2} + 8e^{-1} \\ &= 8(e^{-1} - e^{-2}).\end{aligned}$$

QUESTION A4. Consider the rational function

$$f(x) = \frac{x^2 - 4x}{x^2 - 3x + 2}.$$

(a) Decompose f as a sum of proper fractions.
(b) Use this to determine the indefinite integral $\int f(x)\, dx$.

(a) We do long division:

$$\begin{array}{r|r} & 1 \\ \hline x^2 - 3x + 2 & x^2 - 4x \\ & -x^2 + 3x - 2 \\ \hline & -x - 2 \end{array}$$

which says that

$$\frac{x^2 - 4x}{x^2 - 3x + 2} = 1 + \frac{-x - 2}{x^2 - 3x + 2}.$$

Now we factor the denominator $x^2 - 3x + 2 = (x - 1)(x - 2)$ and solve

$$\frac{-x - 2}{x^2 - 3x + 2} = \frac{A}{x - 1} + \frac{B}{x - 2}.$$

Multiplying this out gives the equation

$$\frac{-x - 2}{x^2 - 3x + 2} = \frac{A(x - 2) + B(x - 1)}{x^2 - 3x + 2}$$

so by considering the numerators alone, plugging in $x = 2$ gives $B = -4$ and $x = 1$ gives $-A = -3$ or $A = 3$. Thus we have

$$\frac{x^2 - 4x}{x^2 - 3x + 2} = 1 + \frac{3}{x - 1} + \frac{-4}{x - 2}.$$

(b) We integrate

$$\begin{aligned} \int \frac{x^2 - 4x}{x^2 - 3x + 2}\, dx &= \int \left(1 + \frac{3}{x - 1} + \frac{-4}{x - 2}\right)\, dx \\ &= x + 3 \ln|x - 1| - 4 \ln|x - 2| + c. \end{aligned}$$

QUESTION A5. The dynamics of a population of fish in a lake follows a logistic model, to which we have added a negative factor owing to predation; this gives the following model

$$\frac{dN}{dt} = 0.15N(400 - N) - 0.3N$$

where $N(t)$ is the number of fish at time t, and t is measured in weeks.
(a) Find all biologically relevant equilibria.
(b) State and apply the Stability theorem to determine the stability of each of the equilibria.
(c) Draw the phase line diagram for this model.

(a) We have to solve the equation $0.15N(400-N)-0.3N = 0$. Clearly $N = 0$ is a solution. To find others we divide by N to get

$$0.15(400 - N) - 0.3 = 0 \Leftrightarrow 60 - 0.15N - 0.3 = 0 \Leftrightarrow 0.15N = 59.7 \Leftrightarrow N = 398.$$

So there are two equilibria, 0 and 398; both are biologically relevant.

(b) The Stability Theorem states that if $N' = f(N)$ is a differential equation and N^* is an equilibrium, then (a) if $f'(N^*) < 0$ then N^* is stable and (b) if $f'(N^*) > 0$ then N^* is unstable.

Here, $f(N) = 0.15N(400 - N) - 0.3N$ so

$$f'(N) = 0.15(400 - N) - 0.15N - 0.3 = 59.7 - 0.3N.$$

Since $f'(0) = 59.7 > 0$, the equilibrium $N^* = 0$ is unstable. Since $f'(398) = -59.7 < 0$, the equilibrium $N^* = 398$ is stable.

(c) The points on the phase line are 0 and 398. Since this is a population model, we only draw the positive phase line, because negative values of N are not meaningful. Between 0 and 398, we see that $f(N) = 0.15N(400 - N) - 0.3N > 0$ (by plugging in $N = 1$, for example), so the arrow points to the right. For $N > 398$ we see that $f(N) < 0$ (by plugging in $N = 1000$, for example); so the arrow points to the left.

(insert picture here)

QUESTION A6. Consider the following integral:

$$\int_0^2 \frac{2xdx}{(x^2-1)^{1/3}}$$

(a) Explain why this integral is improper.
(b) Study its convergence or divergence. If it converges give its value.

(a) The integrand is undefined at $x = \pm 1$. In particular it is not defined on the entire interval $[0, 2]$ since it has a vertical asymptote at $x = 1$, and is thus an improper integral.

(b) Let us first find the indefinite integral. We use the substitution $u = x^2 - 1$ so $du = 2x$. Then

$$\int \frac{2x\,dx}{(x^2-1)^{1/3}} = \int \frac{1}{u^{1/3}}\,du = \int u^{-1/3}\,du = \frac{3}{2}u^{2/3} + c = \frac{3}{2}(x^2-1)^{2/3} + c.$$

Therefore

$$\begin{aligned}
\lim_{T\to 1^-} \int_0^T \frac{2x\,dx}{(x^2-1)^{1/3}} &= \lim_{T\to 1^-} \left[\frac{3}{2}(x^2-1)^{2/3}\right]_0^T \\
&= \lim_{T\to 1^-} \left(\frac{3}{2}(T^2-1)^{2/3} - \frac{3}{2}(-1)\right) \\
&= \frac{3}{2}(1^2-1)^{2/3} - \frac{3}{2}(-1) \\
&= \frac{3}{2}
\end{aligned}$$

which implies that the integral $\int_0^1 \frac{2x\,dx}{(x^2-1)^{1/3}}\,dx$ converges, to $\frac{3}{2}$.

We next check

$$\begin{aligned}
\lim_{T\to 1^+} \int_T^0 \frac{2x\,dx}{(x^2-1)^{1/3}} &= \lim_{T\to 1^+} \left[\frac{3}{2}(x^2-1)^{2/3}\right]_T^2 \\
&= \lim_{T\to 1^+} \left(\frac{3}{2}(3)^{2/3} - \frac{3}{2}(T^2-1)^{2/3}\right) \\
&= \frac{3}{2}3^{2/3} - \frac{3}{2}(1^2-1)^{2/3} \\
&= \frac{3}{2}3^{2/3}
\end{aligned}$$

which implies that the integral $\int_1^2 \frac{2x\,dx}{(x^2-1)^{1/3}}\,dx$ also converges, to $\frac{3}{2}3^{2/3}$.

Therefore the integral $\int_0^2 \frac{2x\,dx}{(x^2-1)^{1/3}}\,dx$ converges to

$$\frac{3}{2} + \frac{3}{2}3^{2/3}.$$

QUESTION A7. (a) Solve the following initial value problem:

$$y' = \frac{xy\sin(x)}{y+1}$$

where $y(0) = 1$. You may leave your answer as an implicit equation of x and y.

This is a separable differential equation.

$$\frac{y+1}{y}y' = x\sin(x)$$
$$\int \frac{y+1}{y}\,dy = \int x\sin(x)\,dx$$
$$\int \left(1+\frac{1}{y}\right)\,dy = \int x\sin(x)\,dx.$$

We solved the integral on the right in Question A1(f) so it is not recopied here. We get

$$y + \ln|y| = \sin(x) - x\cos(x) + c.$$

The initial value gives the equation

$$1 + \ln(1) = \sin(0) - 0\cos(0) + c \Leftrightarrow c = 1.$$

So our solution is

$$y + \ln|y| = \sin(x) - x\cos(x) + 1.$$

In this case we cannot solve for y, so the solution is defined implicitly.

(b) Solve the following initial value problem:

$$x' = x\sec^2(t)$$

with $x(0) = 5$.

This is a separable differential equation.

$$\frac{1}{x}x' = \sec^2(t)$$
$$\int \frac{1}{x}\,dx = \int \sec^2(t)\,dt$$
$$\ln|x| = \tan(t) + c$$
$$|x| = e^{\tan(t)+c}$$
$$x = \pm e^c e^{\tan(t)} = Ae^{\tan(t)}$$

where A is an arbitrary constant. Then $x(0) = 5$ means $5 = Ae^{\tan(0)} = A$ so the solution is

$$x(t) = 5e^{\tan(t)}.$$

You should solve for the state variable in your solution whenever this is possible.

QUESTION A8. Using Euler's method, estimate the value of $y(1)$ if $y(x)$ is the solution of the differential equation

$$y' = xy - x^2$$

satisfying initial condition $y(0) = 1$. Use a step size of $\Delta x = 0.2$.

We use the formula

$$y_{n+1} = y_n + f(x_n, y_n)\Delta x = y_n + (x_n y_n - x_n^2)\Delta x$$

and make a table starting with initial conditions $x_0 = 0$, $y_0 = 1$.

n	x_n	y_n	$x_n y_n - x_n^2$	$y_n + (x_n y_n - x_n^2)\Delta x$
0	0	1	0	1
1	0.2	1	0.16	1.032
2	0.4	1.032	0.2528	1.08256
3	0.6	1.08256	0.289536	1.1404672
4	0.8	1.1404672	0.27237376	1.194941952
5	1	1.194941952		

so our estimate is $y(1) = 1.194941952$.

QUESTION A9. Consider the following vectors and matrices.

$$A = \begin{bmatrix} 1 & -11 & -9 \\ 3 & 0 & 10 \\ 14 & -2 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & -7 \\ 8 & 5 \\ 0 & 1 \end{bmatrix}, \vec{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 5/3 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 1/3 \\ 3/5 \end{bmatrix}.$$

Calculate the following, if possible. If not possible, state why.

a) $A^T\vec{v} + 2B^T\vec{u}$.

$A^T\vec{v} + 2B^T\vec{u}$ is not defined since $B^T\vec{u}$ is 2×1 and $A^T\vec{v}$ is 3×1.

b) $\vec{w}\vec{v}^T$

$$\vec{v}\vec{w}^T = \begin{bmatrix} 0 & 1/3 & 3/5 \\ 0 & 5/9 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

c) $\vec{v}^T\vec{w}$

We can write a 1×1 matrix as a number.

$\vec{v}^T\vec{w} = 1 \times 0 + 5/3 \times 1/3 + 0 \times 3/5 = 5/9$.

d) $A\vec{u} + 2\vec{v}^T\vec{w}$

$A\vec{u} + 2\vec{v}^T\vec{w}$ is not defined since $A^T\vec{u}$ is a 3×1 vector and $\vec{v}^T\vec{w}$ is 1×1.

e) AB

$$AB = \begin{bmatrix} -86 & -71 \\ 6 & -11 \\ 12 & -108 \end{bmatrix}.$$

f) $B\vec{u}$

$B\vec{u}$ is not defined since B has 2 columns and $\vec{u}$ has 3 rows.

g) BA

BA is not defined since B has 2 columns and A has 3 rows.

h) A^2

$$A^2 = \begin{bmatrix} -158 & 7 & -119 \\ 143 & -53 & -27 \\ 8 & -154 & -146 \end{bmatrix}.$$

i) B^2

B^2 is not defined since it is not square. (The first B is 3×2, and the second is 3×2.)

QUESTION A10. Express the following complex number in polar (Euler) form:

$$z = 1 - 2i.$$

$$|z| = \sqrt{1^2 + (-2)^2} = \sqrt{5} = r.$$

Solve

$$r\cos(\varphi) = 1, \quad r\sin(\varphi) = -2.$$

Therefore

$$\cos(\varphi) = \frac{1}{\sqrt{5}} \implies \varphi = 1.107 \text{ or } -1.107 \text{ radians}.$$

Since $\sin(\varphi) < 0$, we deduce the correct answer is $\varphi = -1.107$ and check that indeed $\sin(\varphi) = -2/\sqrt{5}$.

So the polar form of z is

$$z = \sqrt{5}e^{-1.107i}.$$

QUESTION A11. Given $u = 3 + 2i$, $v = -1 + 4i$, compute $\overline{u}v + \frac{|u|^2}{v}$.

$\overline{u} = 3 - 2i$, $|u|^2 = 3^2 + 2^2 = 13$, $|v|^2 = (-1)^2 + 16 = 17$. So

$$\overline{u}v = (3 - 2i)(-1 + 4i) = -3 + 12i + 2i - 8i^2 = 5 + 14i$$

and

$$\frac{|u|^2}{v} = \frac{|u|^2}{|v|^2}\overline{v} = \frac{13}{17}(-1 - 4i) = -\frac{13}{17} - \frac{52}{17}i.$$

Thus

$$\overline{u}v + \frac{|u|^2}{v} = 5 + 14i + \left(-\frac{13}{17} - \frac{52}{17}i\right) = \frac{72}{17} + \frac{186}{17}i.$$

QUESTION A12. For which value(s) of r is $y = x^r \ln(x)$ a solution of the differential equation

$$y' = \frac{y}{x}\left(\frac{3\ln(x) + 1}{\ln(x)}\right)?$$

If $y = x^r \ln(x)$ then

$$y' = rx^{r-1}\ln(x) + x^r\frac{1}{x} = x^{r-1}(r\ln(x) + 1)$$

whereas

$$\frac{y}{x}\left(\frac{3\ln(x) + 1}{\ln(x)}\right) = \frac{x^r\ln(x)}{x}\left(\frac{3\ln(x) + 1}{\ln(x)}\right) = x^{r-1}(3\ln(x) + 1).$$

Comparing these answers reveals that in order to solve the given differential equation, we must have $r = 3$.