# COMP. 233.

**MIDTERM.** (Total 45 marks)

QUESTION 1. (2 marks)

An urn contains 20 white marbles, and 6 black marbles. Seven marbles are randomly chosen from the urn. What is the probability that no black marbles are among the seven chosen? (All marbles are distinct).

QUESTION 2. (12 marks)

How many positive integers between 1000 and 9999 inclusive:

A. Are divisible by 9?

B. Are even?

C. Have distinct digits?

D. Are not divisible by 3?

QUESTION 3. (3 marks)

How many bit strings of length seven either begin with two 0's or end with three 1's?

QUESTION 4. (8 marks)

How many bit strings of length twelve contain:

A. Exactly three 1's?

B. At most three 1's?

C. At least three 1's?

D. An equal number of 0's and 1's?

QUESTION 5. (3 marks)

How many bit strings contain exactly eight 0's and ten 1's if every 0 must be immediately followed by a 1?

QUESTION 6. (3 marks)

Suppose that one person in 100,000 has a particular rare disease for which there is a fairly accurate diagnostic test. This test is correct 99% of the time when given to someone with the disease; it is correct 99.5% of the time when given to someone who does not have the disease.

What is the probability that someone who tests positive for the disease, has the disease?

QUESTION 7. (3 marks)

Let the R.V. X be Poisson Distributed, with parameter $\lambda = 9.1$.

A. $P(10 \leq X < 19)$?

B. $P(X > 20)$?

C. $P(X < 10)$?

QUESTION 8. (1 mark)

Let the R.V. $X \sim N(20, 4)$.

$P( | X | \leq 16 )$?

QUESTION 9. (2 marks)

Three balls are randomly chosen from an urn containing 3 white, 4 red, and 5 black balls. Suppose one will win $1 for each white ball selected, lose $1 for each red ball selected, and receive $0 for each black ball selected.
Let the R.V. X denote the total winnings from the experiment.

A. P(X = 0)?

B. P(X = 1)?

QUESTION 10. (2 marks)

Three balls are randomly chosen from an urn containing 3 white, 4 red, and 5 black balls.

Let R and W denote, respectively, the number of red and white balls chosen.

Let P(R = i, W = j) be the P.M.F.

A. P(0, 1)?

B. P(1,1)?

QUESTION 11. (3 marks)

A Joint Density Function for X and Y is given by:

$$f(x, y) = \begin{cases} 2e^{-x}\, e^{-2y} & x > 0,\ y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

P(X < a)?

QUESTION 12. (3 marks)

Suppose a fair die is rolled until a 6 comes up.

Let the R.V. X: denote the number of rolls needed.

A. What distribution does X have?

B. E[X]?

C. Var(X)?

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## Midterm Solutions.

1.

$$\frac{{}_{20}C_7}{{}_{26}C_7} = 0.118$$

2.

Range: (9999 - 1000) + 1= 9000 numbers.

A. Every ninth number is divisible by 9.

$\therefore$, $\frac{9000}{9} = 1000$ numbers are divisible by 9.

B. Every second number is even.

$\frac{9000}{2} = 4{,}500$ numbers are even.

C. $(9)(9)(8)(7) = 4{,}536$ numbers have distinct digits.

D. $\frac{9000}{3} = 3000$ numbers are divisible by 3.

$\therefore$, $9000 - 3000 = 6000$ numbers are not divisible by 3.

3.

Let A: Set of bit strings starting with 00.

B: Set of bit strings ending with 111.

C: Set of bit strings starting with 00 or ending with 111.

$$|A| = 2^5 = 32$$
$$|B| = 2^4 = 16$$
$$|C| = |A \cup B| = |A| + |B| - |A \cap B|$$
$$= 32 + 16 - 2^2$$
$$= 44$$

4.

A. $\binom{12}{3} = {}_{12}C_3 = 220$ strings.

B. $\binom{12}{3} + \binom{12}{2} + \binom{12}{1} + \binom{12}{0} = 220 + 66 + 12 + 1 = 299$ strings.

C. $2^{12} - \left[\binom{12}{2} + \binom{12}{1} + \binom{12}{0}\right] = 4017$ strings.

D. ${}_{12}C_6 = 924$ strings.

5.

Bit string length: 18
Number of 01 pairs: 8
$(8)(2) = 16$ bit positions occupied ⇒ the two remaining positions must be occupied by 1's.
Number of grouped bit string positions 8 pairs + 2 singles = 10 bit string positions (grouped bit string length is 10).
∴, $\binom{10}{2} = 45$ strings.

6.

Let H: event person has the disease.
T: event person tests positive for the disease.

$$P(H|T)?$$

$$P(H) = \frac{1}{100000} = 0.00001$$

$$P(\overline{H}) = 0.99999$$

$$P(T|H) = 0.99$$

$$P(\bar{T}|H) = 0.1$$
$$P(\bar{T}|\bar{H}) = 0.995$$
$$P(T|\bar{H}) = 0.005$$

.

$$\therefore, \qquad P(H|T) = \frac{P(HT)}{P(T)}$$
$$= \frac{P(H)\,P(T|H)}{P(T)}$$
$$= \frac{P(H)\,P(T|H)}{P(TH \cup T\bar{H})}$$
$$= \frac{P(H)\,P(T|H)}{P(TH) + P(T\bar{H})}$$
$$= \frac{P(H)\,P(T|H)}{P(H)P(T|H) + P(\bar{H})P(T|\bar{H})}$$
$$= \frac{(0.00001)(0.99)}{(0.00001)(0.99) + (0.99999)(0.005)}$$
$$= 0.002$$

7.

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$X \sim P(9.1)$ (Use tables).

A.

$$P(10 \leq X < 19) = \sum_{i=10}^{18} \frac{e^{-9.1}(9.1)^i}{i!} = 0.423$$

B.

$$P(X > 20) = \sum_{i=21}^{22} \frac{e^{-9.1}(9.1)^i}{i!} = 0.0004$$

C.

$$P(X < 10) = \sum_{i=0}^{9} \frac{e^{-9.1}(9.1)^i}{i!} = 0.5742$$

8.

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$X \sim N(20, 4)$.

$$\begin{aligned} P(|X| \leq 16) &= P(-16 \leq X \leq 16) \\ &= P\left(\frac{-16-20}{2} \leq \frac{X-20}{2} \leq \frac{16-20}{2}\right) \\ &= P(-18 \leq Z \leq -2) \\ &= P(Z \leq 18) - P(Z \leq 2) \\ &= 1 - 0.9772 \\ &= 0.0228 \end{aligned}$$

9.

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A.

$$P(X=0) = \frac{{}_5C_3}{{}_{12}C_3} + \frac{({}_3C_1)({}_4C_1)({}_5C_1)}{{}_{12}C_3}$$

$$= 0.3\overline{18}$$

$$P(X=0) = P(3\ black\ balls) + P(1\ white, 1\ red, 1\ black)$$

B.

$$P(X=1) = P(1W, 2B) + P(2W, 1R)$$

$$= \frac{({}_3C_1)({}_5C_2)}{{}_{12}C_3} + \frac{({}_3C_2)({}_4C_1)}{{}_{12}C_3}$$

$$= 0.19$$

10.

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A.

$$P(0,1) = P(R = 0, W = 1, B = 2)$$

$$= \frac{({}_3C_1)({}_4C_0)({}_5C_2)}{{}_{12}C_3}$$

$$= 0.1\overline{36}$$

B.

$$P(1,1) = P(R = 1, W = 1, B = 1)$$

$$= \frac{({}_4C_1)({}_3C_1)({}_5C_1)}{{}_{12}C_3}$$

$$= 0.\overline{27}$$

11.

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$$P(X < a) = \int_0^a \int_0^\infty 2e^{-x}e^{-2y}\, dydx$$

$$P(X < a) = \int_0^a e^{-x}\, dx = 1 - e^{-a}$$

12.

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A. X has a Geometric Distribution with parameter: $p = \frac{1}{6}$.

B. $E[X] = \frac{1}{p} = 6$ (mean of a Geometric Distribution is $\frac{1}{p}$).

C. $Var(X) = \frac{1-p}{p^2} = \frac{(1-\frac{1}{6})}{\frac{1}{36}} = 30$ (variance of a Geometric Distribution is $\frac{1-p}{p^2}$).