University of Ottawa
School of Electrical Engineering and Computer Science
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Assignment #1 Solution

46. The truth value of "Fred and John are happy" is $\min(0.8, 0.4) = 0.4$. The truth value of "Neither Fred nor John is happy" is $\min(0.2, 0.6) = 0.2$, since this statement means "Fred is not happy, and John is not happy," and we computed the truth values of the two propositions in this conjunction in Exercise 45.

18. We will translate these conditions into statements in symbolic logic, using j, s, and k for the propositions that Jasmine, Samir, and Kanti attend, respectively. The first statement is $j \to \neg s$. The second statement is $s \to k$. The last statement is $\neg k \vee j$, because"unless" means "or." (We could also translate this as $k \to j$. From the comments following Definition 5 in the text, we know that $p \to q$ is equivalent to "q unless $\neg p$. In this case p is $\neg j$ and q is $\neg k$.) First, suppose that s is true. Then the second statement tells us that k is also true, and then the last statement forces j to be true. But now the first statement forces s to be false. So we conclude that s must be false; Samir cannot attend. On the other hand, if s is false, then the first two statements are automatically true, not matter what the truth values of k and j are. If we look at the last statement, we see that it will be true as long as it is not the case that k is true and j is false. So the only combinations of friends that make everybody happy are Jasmine and Kanti, or Jasmine alone (or no one!).

12. We argue directly by showing that if the hypothesis is true, then so is the conclusion. An alternative approach, which we show only for part **(a)**, is to use the equivalences listed in the section and work symbolically.

a) Assume the hypothesis is true. Then p is false. Since $p \vee q$ is true, we conclude that q must be true. Here is a more "algebraic" solution: $[\neg p \wedge (p \vee q)] \to q \equiv \neg[\neg p \wedge (p \vee q)] \vee q \equiv \neg\neg p \vee \neg(p \vee q)] \vee q \equiv p \vee \neg(p \vee q) \vee q \equiv (p \vee q) \vee \neg(p \vee q) \equiv \mathbf{T}$. The reasons for these logical equivalences are, respectively, Table 7, line 1; De Morgan's law; double negation; commutative and associative laws; negation law.

b) We want to show that if the entire hypothesis is true, then the conclusion $p \to r$ is true. To do this, we need only show that if p is true, then r is true. Suppose p is true. Then by the first part of the hypothesis, we conclude that q is true. It now follows from the second part of the hypothesis that r is true, as desired.

c) Assume the hypothesis is true. Then p is true, and since the second part of the hypothesis is true, we conclude that q is also true, as desired.

d) Assume the hypothesis is true. Since the first part of the hypothesis is true, we know that either p or q is true. If p is true, then the second part of the hypothesis tells us that r is true; similarly, if q is true, then the third part of the hypothesis tells us that r is true. Thus in either case we conclude that r is true.

24. We determine exactly which rows of the truth table will have T as their entries. Now $(p \to q) \vee (p \to r)$ will be true when either of the conditional statements is true. The conditional statement will be true if p is false, or if q in one case or r in the other case is true, i.e., when $q \vee r$ is true, which is precisely when $p \to (q \vee r)$ is true. Since the two propositions are true in exactly the same situations, they are logically equivalent.

10. a) We assume that this means that one student has all three animals: $\exists x(C(x) \wedge D(x) \wedge F(x))$.
b) $\forall x(C(x) \vee D(x) \vee F(x))$ **c)** $\exists x(C(x) \wedge F(x) \wedge \neg D(x))$
d) This is the negation of part **(a)**: $\neg \exists x(C(x) \wedge D(x) \wedge F(x))$.
e) Here the owners of these pets can be different: $(\exists x\, C(x)) \wedge (\exists x\, D(x)) \wedge (\exists x\, F(x))$. There is no harm in using the same dummy variable, but this could also be written, for example, as $(\exists x\, C(x)) \wedge (\exists y\, D(y)) \wedge (\exists z\, F(z))$.

42. There are many ways to write these, depending on what we use for predicates.
a) Let $A(x)$ be "User x has access to an electronic mailbox." Then we have $\forall x\, A(x)$.
b) Let $A(x, y)$ be "Group member x can access resource y," and let $S(x, y)$ be "System x is in state y." Then we have $S(\text{file system}, \text{locked}) \to \forall x\, A(x, \text{system mailbox})$.
c) Let $S(x, y)$ be "System x is in state y." Recalling that "only if" indicates a necessary condition, we have $S(\text{firewall}, \text{diagnostic}) \to S(\text{proxy server}, \text{diagnostic})$.
d) Let $T(x)$ be "The throughput is at least x kbps," where the domain of discourse is positive numbers, let $M(x, y)$ be "Resource x is in mode y," and let $S(x, y)$ be "Router x is in state y." Then we have $(T(100) \wedge \neg T(500) \wedge \neg M(\text{proxy server}, \text{diagnostic})) \to \exists x\, S(x, \text{normal})$.

46. a) There are two cases. If A is true, then $(\forall x P(x)) \vee A$ is true, and since $P(x) \vee A$ is true for all x, $\forall x(P(x) \vee A)$ is also true. Thus both sides of the logical equivalence are true (hence equivalent). Now suppose that A is false. If $P(x)$ is true for all x, then the left-hand side is true. Furthermore, the right-hand side is also true (since $P(x) \vee A$ is true for all x). On the other hand, if $P(x)$ is false for some x, then both sides are false. Therefore again the two sides are logically equivalent.
b) There are two cases. If A is true, then $(\exists x P(x)) \vee A$ is true, and since $P(x) \vee A$ is true for some (really all) x, $\exists x(P(x) \vee A)$ is also true. Thus both sides of the logical equivalence are true (hence equivalent). Now suppose that A is false. If $P(x)$ is true for at least one x, then the left-hand side is true. Furthermore, the right-hand side is also true (since $P(x) \vee A$ is true for that x). On the other hand, if $P(x)$ is false for all x, then both sides are false. Therefore again the two sides are logically equivalent.

20. a) $\forall x \forall y((x < 0) \wedge (y < 0) \to (xy > 0))$ **b)** $\forall x \forall y((x > 0) \wedge (y > 0) \to ((x + y)/2 > 0))$
c) What does "necessarily" mean in this context? The best explanation is to assert that a certain universal conditional statement is not true. So we have $\neg \forall x \forall y((x < 0) \wedge (y < 0) \to (x - y < 0))$. Note that we do not want to put the negation symbol inside (it is not true that the difference of two negative integers is never negative), nor do we want to negate just the conclusion (it is not true that the sum is always nonnegative). We could rewrite our solution by passing the negation inside, obtaining $\exists x \exists y((x < 0) \wedge (y < 0) \wedge (x - y \geq 0))$.
d) $\forall x \forall y\, (|x + y| \leq |x| + |y|)$

32. As we push the negation symbol toward the inside, each quantifier it passes must change its type. For logical connectives we either use De Morgan's laws or recall that $\neg(p \to q) \equiv p \wedge \neg q$ (Table 7 in Section 1.3) and that $\neg(p \leftrightarrow q) \equiv \neg p \leftrightarrow q$ (Exercise 21 in Section 1.3).

a)
$$\begin{aligned}\neg\exists z\forall y\forall x\, T(x,y,z) &\equiv \forall z\neg\forall y\forall x\, T(x,y,z)\\ &\equiv \forall z\exists y\neg\forall x\, T(x,y,z)\\ &\equiv \forall z\exists y\exists x\, \neg T(x,y,z)\end{aligned}$$

b)
$$\begin{aligned}\neg(\exists x\exists y\, P(x,y) \wedge \forall x\forall y\, Q(x,y)) &\equiv \neg\exists x\exists y\, P(x,y) \vee \neg\forall x\forall y\, Q(x,y)\\ &\equiv \forall x\neg\exists y\, P(x,y) \vee \exists x\neg\forall y\, Q(x,y)\\ &\equiv \forall x\forall y\, \neg P(x,y) \vee \exists x\exists y\, \neg Q(x,y)\end{aligned}$$

c)
$$\begin{aligned}\neg\exists x\exists y(Q(x,y) \leftrightarrow Q(y,x)) &\equiv \forall x\neg\exists y(Q(x,y) \leftrightarrow Q(y,x))\\ &\equiv \forall x\forall y\neg(Q(x,y) \leftrightarrow Q(y,x))\\ &\equiv \forall x\forall y(\neg Q(x,y) \leftrightarrow Q(y,x))\end{aligned}$$

d)
$$\begin{aligned}\neg\forall y\exists x\exists z\,(T(x,y,z) \vee Q(x,y)) &\equiv \exists y\neg\exists x\exists z\,(T(x,y,z) \vee Q(x,y))\\ &\equiv \exists y\forall x\neg\exists z\,(T(x,y,z) \vee Q(x,y))\\ &\equiv \exists y\forall x\forall z\,\neg(T(x,y,z) \vee Q(x,y))\\ &\equiv \exists y\forall x\forall z\,(\neg T(x,y,z) \wedge \neg Q(x,y))\end{aligned}$$

46. This statement says that there is a number that is less than or equal to all squares.

a) This is false, since no matter how small a positive number x we might choose, if we let $y = \sqrt{x/2}$, then $x = 2y^2$, and it will not be true that $x \le y^2$.

b) This is true, since we can take $x = -1$, for example.

c) This is true, since we can take $x = -1$, for example.