

**MIDTERM TEST SAMPLE SOLUTION**

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**PART A**

**1. (4 points) What are truncation errors? How did they get their name?**

(2 points) Truncation errors are errors produced by numerical methods.

(2 points) Their name originates from the fact that many numerical methods are derived from Taylor series that are truncated after different number of terms.

**2. (4 points) What is the key pitfall of the Secant Method?**

May shoot off to find a different root from expected if the function is not linear near the root.

**3. (4 points) What method is least sensitive to ill-conditioned matrices? Explain.**

(2 points) The iterative techniques such as Gauss-Seidel and the Jacobi method are the least sensitive to ill-conditioned matrices

(2 points) The iterative techniques do carry over round-off errors from one iteration to the next. While the direct solution methods each step adds to the round-off errors of the previous method thus making the solution more sensitive to the small changes in the initial values of the original matrix coefficients.

**4. (4 points) How do the order of the method (i.e., the theoretical rate of convergence) differ from the actual rate of convergence?**

The theoretical rate of convergence is the maximum rate of conversion of a given numerical methods.

The calculations performed in the lab, and reported as part of assignment#3, clearly showed that the actual rate of convergence varied with the function and the selection of the initial guesses. So they may reach the maximum rate of conversion but they will not do so in many situations. In addition the rate of conversion may actually decrease as the technique approaches the root.

**5. ( 4 points) We have two systems of two linear equations following:**

$$\begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{2} & 1 \end{pmatrix} * \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$$
$$\begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & 2 \end{pmatrix} * \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

**Explain why these two systems are singular.**

Assume the first matrix is [R] and the second matrix is [T]

$$\det[R] = \left(-\frac{1}{2} \times 1\right) - \left(1 \times \left(-\frac{1}{2}\right)\right) = 0$$

$$\det[T] = \left(-\frac{1}{2} \times 2\right) - (1 \times (-1)) = 0$$

Since the determinant of both these matrices equal zero they are singular matrices.

**6. ( 4 points) Two coefficient matrices A and B are given below::**

$$A = \begin{bmatrix} a1 & a2 & 0 & 0 & 0 & 0 \\ b1 & b2 & b3 & 0 & 0 & 0 \\ 0 & c1 & c2 & c3 & 0 & 0 \\ 0 & 0 & d1 & d2 & d3 & 0 \\ 0 & 0 & 0 & e1 & e2 & e3 \\ 0 & 0 & 0 & 0 & f1 & f2 \end{bmatrix} \quad B = \begin{bmatrix} a1 & a2 & a3 & a4 & a5 & a6 \\ b1 & b2 & b3 & b4 & b5 & b6 \\ c1 & c2 & c3 & c4 & c5 & c6 \\ d1 & d2 & d3 & d4 & d5 & d6 \\ e1 & e2 & e3 & e4 & e5 & e6 \\ f1 & f2 & f3 & f4 & f5 & f6 \end{bmatrix}$$

**What is the advantage of solving matrix A over matrix B, in terms of computational cost?**

Matrix A is a tri-diagonal matrix (i.e., it only contains zero coefficients) outside the main three diagonals. This type of matrices can be solved by the Thomas method which is much more computational efficient in that: a) has a lower storage requirement; and b) it requires less mathematical operations.

**7. ( 4 points) If  $x=g(x)$  converges, explain why  $x=g(x)\sin(x)$  also converges.**

If  $x=g(x)$  converges, then  $|g'(x)| < 1$ .

Call the new function  $gg = g(x)*\sin(x)$  and  $gg'(x) = g'(x)*\sin(x) + g(x)*\cos(x)$

However, if  $g(x)=0.5*x$  and the evaluation is at  $x=2\pi$ .

$gg'(2\pi) = (0.5)*(\sin(2\pi)) + (0.5*2\pi)*(\cos(2\pi)) = (0.5*0) + (\pi*1) = \pi$  so in this case it diverges.

## **PART B**

**B.1 (20 points) Use the Regula Falsi (False Root) method to solve the following equation:**

$$\sin(x) = x^3$$

### **TASKS:**

**a) Using only four decimal places conduct two iterations using the false root method. Show every step in total detail. Describe why each point is selected for every iteration.**

**Use  $x=0.5$  and  $x=1.5$  as initial guesses.  $f(0.5)= 0.3544$  and  $f(1.5)= -2.3775$**

$$f(x) = \sin(x) - x^3$$

The Regula Falsi equation is

$$x_3 = x_2 - \frac{f(x_2)}{f(x_1) - f(x_2)}(x_1 - x_2) = 0.61297$$

$$x_1=0.5 \text{ and } x_2=1.5$$

#### First iteration

$$x_3 = 1.5 - \frac{-2.3775}{0.3544 - (-2.3775)}(0.5 - 1.5) = 0.6297 = 0.61297$$

$$f(x_3) = \sin(0.6297) - (0.6297)^3 = 0.3392$$

To keep the bracketing (i.e.  $f(x_1)*f(x_2)<0$ ), then  $x_3$  should replace  $x_1$  in the next iteration

#### Second iteration

$$x_1=0.6297, x_2=1.5, f(x_1) = 0.3392, \text{ and } f(x_2) = -2.3775$$

$$x_3 = 1.5 - \frac{-2.3775}{0.3392 - (-2.3775)}(0.6297 - 1.5) = 0.7384 = 0.61297$$

$$f(x_3) = \sin(0.7384) - (0.7384)^3 = 0.2705$$

**b) What would be the points and function values to be used in the third iteration?**

To keep the bracketing (i.e.  $f(x_1)*f(x_2)<0$ ), then  $x_3$  should again replace  $x_1$  in the third iteration, so the points and the function evaluations will be

$$x_1=0.7384, x_2=1.5, f(x_1) = 0.2705, \text{ and } f(x_2) = -2.3775$$

B.2. (10 points) In trying to find the root of given non-linear equation yields the numbers below. The real root of the equation is 10.

Iteration #	$x_i$
1	5.21
2	5.6
3	6.29
4	7.36
5	8.66
6	9.65

**Find:**

a) **The rate of convergence**

The rate of convergence is calculate from the plot of  $\ln E_{i+1}$  versus  $\ln E_i$  where  $E_i = |x^* - x_i|$

So for iteration 1,  $E_1 = |10 - 5.21| = 4.79$  and  $\ln E_1 = 1.5665$

Repeating this for all the iterations one gets

$x$	$E_i$ ( $x^* - x_i$ )	$\ln(E_i)$
5.21	4.79	1.5665304
5.60	4.4	1.4816045
6.29	3.71	1.3110319
7.36	2.64	0.9707789
8.66	1.34	0.2926696
9.65	0.35	-1.049822

The problem is not specific as to whether the overall rate of convergence is required or it can be the rate of convergence from three consecutive points.

Based on rate of convergence using three consecutive points one gets the table below. As an example calculation using the first data points one gets:

$$r_i = \frac{\ln E_3 - \ln E_2}{\ln E_2 - \ln E_1} = \frac{1.3110319 - (1.4816045)}{1.4816045 - (1.5665304)} = 2.0085$$

<b>X</b>	<b>E<sub>i</sub> (X* - X<sub>i</sub>)</b>	<b>ln(E<sub>i</sub>)</b>	<b>r</b>
5.21	4.79	1.5665304	
5.60	4.4	1.4816045	
6.29	3.71	1.3110319	2.008488854
7.36	2.64	0.9707789	1.994768393
8.66	1.34	0.2926696	1.992956371
9.65	0.35	-1.049822	1.979757146

**b) Based on this rate estimate which method generated these iterations. EXPLAIN WHY?**

Since the rate of convergence is 2, the numerical technique must be the Newton-Raphson technique or possibly a second order Newton's method.

**B-3. (10 points) Is the matrix below ill-conditioned? Why or why not? Explain in detail and show all the necessary calculations.**

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & -1.5 & 0.5 \\ -1 & 2 & -1 \end{bmatrix} \quad X = \begin{bmatrix} -4 \\ 3.5 \\ 1 \end{bmatrix}$$

In order to determine matrix A is ill-

conditioned, calculate the condition number.

$$\text{Cond \#} = \|A\| \cdot \|A^{-1}\|$$

Calculate the norms A and  $A^{-1}$ , i.e.,  $\|A\|$  and  $\|A^{-1}\|$

Using the Euclidian norms

$$\|A\| = \sqrt{\prod_{i=1, j=1}^{i=3, j=3} a_{i,j}^2} = \sqrt{(1)^2 + (2)^2 + (2)^2 + (3)^2 + (4)^2 + (5)^2 + (5)^2 + (6)^2 + (7)^2} = 13$$

$$\begin{aligned} \|A^{-1}\| &= \sqrt{\prod_{i=1, j=1}^{i=3, j=3} (a_{i,j}^{-1})^2} \\ &= \sqrt{(-1)^2 + (-1)^2 + (1)^2 + (2)^2 + (-1.5)^2 + (0.5)^2 + (-1)^2 + (2)^2 + (-1)^2} \\ &= 3.9370 \end{aligned}$$

$$\text{Cond \#} = \|A\| \cdot \|A^{-1}\| = \|13\| \cdot \|3.937\| = 51.1811$$

This is relatively large which indicates matrix A is ill-conditioned.

Using alternative norms yields:

#### Maximum Column Norms

norm of A=	14
norm of inverse of A=	4.5
Condition number =	63

#### Maximum Row Norms

norm of A=	18
norm of inverse of A=	4
Condition number =	72

So regardless of the norm used the condition number is quite large and indicative of ill-conditioning.

$$f(x_{i+1}) = f(x_i) + \frac{f'(x_i)}{1!} \cdot h + \frac{f''(x_i)}{2!} \cdot h^2 + \frac{f'''(x_i)}{3!} \cdot h^3 + \dots + \frac{f^{(n)}(x_i)}{n!} \cdot h^n + R_n$$

**B.4 (32 points)**

a) Solve the equations below using the LU factorization approach.

$$\begin{aligned} 4x_1 - 2x_2 + 7x_3 &= 4 \\ -2x_1 + 6x_2 - 8x_3 &= -6 \\ 7x_1 - 8x_2 + 5x_3 &= -0.7 \end{aligned}$$

b) Is this the most suitable approach for these equations? Explain why or why not.

**SHOW ALL YOUR CALCULATIONS IN DETAIL OR LOOSE MARKS  
CARRY ALL CALCULATIONS USING AT LEAST FOUR DECIMAL PLACES**

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} = \begin{bmatrix} 4 & -2 & 7 \\ -2 & 6 & -8 \\ 7 & -8 & 5 \end{bmatrix}$$

Proceed with the Crout LU decomposition

First, calculate the first column of L

$$\begin{aligned} l_{1,1} &= a_{1,1} = 4 \\ l_{2,1} &= a_{2,1} = -2 \\ l_{3,1} &= a_{3,1} = 7 \end{aligned}$$

Then calculate the first row of U

$$\begin{aligned} u_{1,1} &= 1 \\ u_{1,2} &= \frac{a_{2,1}}{l_{1,1}} = \frac{-2}{4} = -0.5 \\ u_{1,3} &= \frac{a_{3,1}}{l_{1,1}} = \frac{7}{4} = 1.75 \end{aligned}$$

Now calculate the second column of L

$$\begin{aligned} l_{2,2} &= a_{2,2} - (l_{2,1} \cdot u_{1,2}) = 6 - ((-2) \cdot (-0.5)) = 5 \\ l_{3,2} &= a_{3,2} - (l_{3,1} \cdot u_{1,2}) = -8 - ((7) \cdot (-0.5)) = -4.5 \end{aligned}$$

Then calculate the second row of U

$$u_{2,3} = \frac{a_{2,3} - ((l_{2,1}) \cdot (u_{1,3}))}{l_{2,2}} = \frac{-8 - ((-2) \cdot (1.75))}{5} = -0.9$$

Now calculate the second column of L

$$l_{3,3} = a_{3,3} - (l_{3,1} \cdot u_{1,3}) - (l_{3,2} \cdot u_{2,3}) = 5 - ((7) \cdot (1.75)) - ((-4.5) \cdot (-0.9)) = -11.3$$

So

$$L = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 5 & 0 \\ 7 & -4.5 & -11.3 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & -0.5 & 1.75 \\ 0 & 1 & -0.9 \\ 0 & 0 & 1 \end{bmatrix}$$

To solve for x, first go through the forward substitution to solve for Z, that is [L][Z]=[b]

$$\begin{bmatrix} 4 & 0 & 0 \\ -2 & 5 & 0 \\ 7 & -4.5 & -11.3 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -0.7 \end{bmatrix}$$

$$4z_1 = 4 \quad \text{so } z_1 = 1$$

$$-2z_1 + 5z_2 = -6 \quad \text{so } z_2 = \frac{(-6 - (-2 \cdot (1)))}{5} = -0.8$$

$$7z_1 - 4.5z_2 - 11.3z_3 = -0.7 \quad \text{so } z_3 = \frac{(-0.7 - (7 \cdot (1)) - (-4.5 \cdot (-0.8)))}{-11.3} = 1$$

$$\text{so } \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.8 \\ 1 \end{bmatrix}$$

Now proceed to the back substitution to solve for x, that is [U][x]=[Z]

$$\begin{bmatrix} 1 & -0.5 & 1.75 \\ 0 & 1 & -0.9 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.8 \\ 1 \end{bmatrix}$$

So

$$1x_3 = 1 \quad \text{so } x_3 = 1$$

$$1x_2 - 0.9x_3 = -0.8 \quad \text{so } x_2 = \frac{-0.8 - (-0.9 \cdot 1)}{1} = 0.1$$

$$1x_1 - 0.5x_2 + 1.75x_3 = 1 \quad \text{so } x_1 = \frac{1 - (1.75 \cdot 1) - (-0.5 \cdot 0.1)}{1} = -0.7$$

$$\text{so the answer is } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.7 \\ 0.1 \\ 1 \end{bmatrix}$$

To verify this is correct substitute back into the equations

$$\begin{aligned} 4x_1 - 2x_2 + 7x_3 &= 4 \cdot (-0.7) - 2 \cdot (0.1) + 7 \cdot (1) = 4 \quad \text{CHECKS} \\ -2x_1 + 6x_2 - 8x_3 &= -2 \cdot (-0.7) + 6 \cdot (0.1) - 8 \cdot (1) = -6 \quad \text{CHECKS} \\ 7x_1 - 8x_2 + 5x_3 &= 7 \cdot (-0.7) - 8 \cdot (0.1) + 5 \cdot (1) = -0.7 \quad \text{CHECKS} \end{aligned}$$