CVG2181 – Winter 201 9

Assignment #2 – Solutions

QUESTION 1

a)

f(x)=(x-r)^m h(x), f(r)=0 we have to prove that u(r)=0

$$\boldsymbol{u(x) = \frac{f(x)}{f'(x)}}$$

$$\boldsymbol{u(x) = \frac{(x-r)^m h(x)}{m(x-r)^{m-1}h(x) + \ (x-r)^m h'(x)} = \frac{(x-r)h(x)}{mh(x) + (x-r)h'(x)}}$$

$$\to u(r) = \boldsymbol{\frac{(r-r)h(r)}{mh(r) + (r-r)h'(r)} = \frac{0 \times h(r)}{m \times h(r) + 0 \times h'(r)} = 0}$$

b)

We have to prove that $\boldsymbol{u'(r) \neq 0}$

So

$$\boldsymbol{u'(x) = \frac{[h(x) + (x-r)h'(x)][mh(x) + (x-r)h'(x)]}{[mh(x) + (x-r)h'(x)]^2} - \frac{[(x-r)h(x)][mh'(x) + h'(x) + (x-r)h''(x)]}{[mh(x) + (x-r)h'(x)]^2}}$$

$$\boldsymbol{u'(r) = \frac{[h(r) + (r-r)h'(x)][mh(r) + (r-r)h'(r)]}{[mh(r) + (r-r)h'(x)]^2} - \frac{[(r-r)h(x)][mh'(r) + h'(r) + (r-r)h''(r)]}{[mh(r) + (r-r)h'(r)]^2}}$$

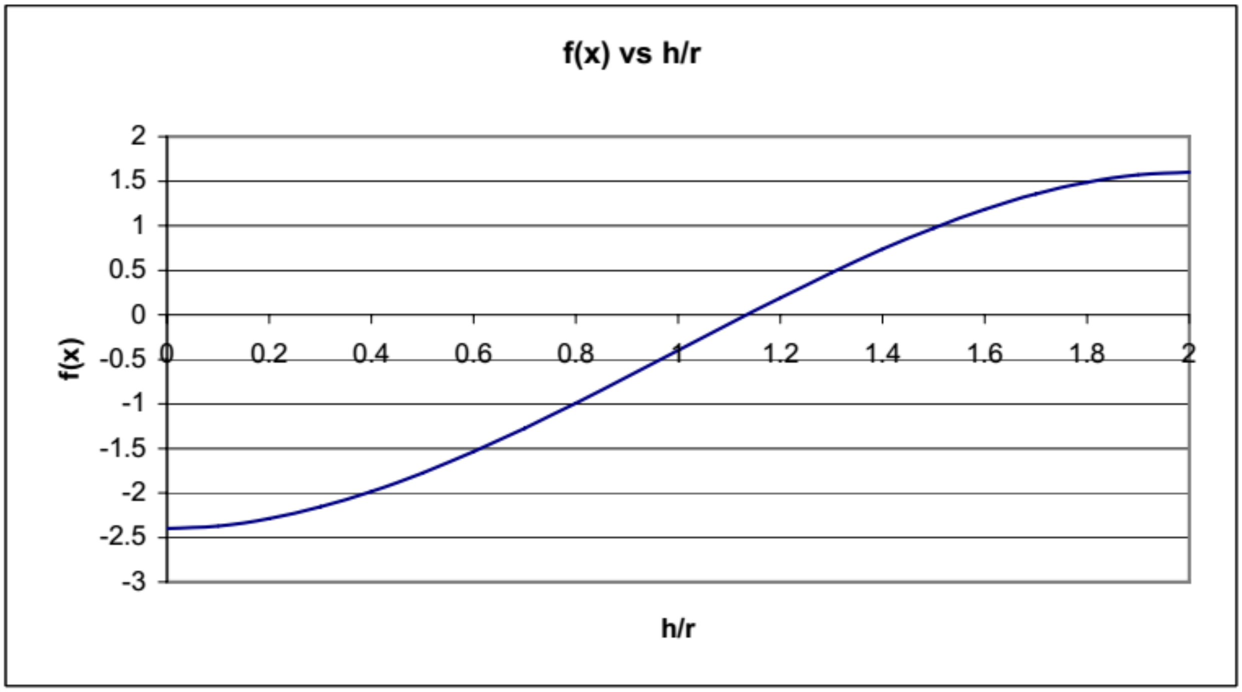
$$\boldsymbol{u'(r) = \frac{[h(r) + (0)h'(x)][mh(r) + (r-r)h'(r)]}{[mh(r) + (0)h'(x)]^2} - \frac{[(0)h(x)][mh'(r) + h'(r) + (r-r)h''(r)]}{[mh(r) + (0)h'(r)]^2}}$$

$$\boldsymbol{u'(r) = \frac{h(r)}{mh(r)} = \frac{1}{m} \neq 0}$$

QUESTION 2

$$f(x) = 3\left(\frac{h}{r}\right)^2 - \left(\frac{h}{r}\right)^3 - 2.4$$

a)



<u>First iteration</u>

$x_1 = 0$

$x_2 = 2$

$$f(x) = 3\left(\frac{h}{r}\right)^2 - \left(\frac{h}{r}\right)^3 - 2.4$$

$$f(x_1) = 3(0)^2 - (0)^3 - 2.4$$
$$= -2.4$$

$$f(x_2) = 3(2)^2 - (2)^3 - 2.4$$
$$= 1.6$$

The Regula Falsi method:

$$x_3 = x_2 - \frac{f(x_2)}{\left[f(x_2) - f(x_1)\right]}(x_2 - x_1)$$

$$x_3 = 2 - \frac{1.6}{\left[1.6 - (-2.4)\right]}(2 - 0)$$
$$= 1.2$$

$$f(x_3) = 3(1.2)^2 - (1.2)^3 - 2.4$$
$$= 0.192$$

Because $f(x_3) > 0$, x_3 replaces x_2

<u>Second iteration</u>

$x_1 = 0$
$x_2 = 1.2$

$f(x_1) = - 2.4$
$f(x_2) = 0.192$

$$x_3 = 1.2 - \frac{0.192}{\left[0.192 - (-2.4)\right]}(1.2 - 0)$$
$$= 1.1111$$

$$f(x_3) = 3(1.1111)^2 - (1.1111)^3 - 2.4$$
$$= 0 - 0.0681$$

Because $f(x_3) < 0$, x_3 replaces x_1

<u>Third iteration</u>

$x_1 = 1.1111$
$x_2 = 1.2$

$f(x_1) = - 0.0681$
$f(x_2) = 0.192$

$$x_3 = 1.2 - \frac{0.192}{\left[0.192 - (-0.0681)\right]}(1.2 - 1.1111)$$
$$= 1.2 - (0.7382)(0.0889)$$
$$= 1.1344$$

$$f(x_3) = 3(1.1344)^2 - (1.1344)^3 - 2.4$$
$$= 0.00077$$

Because $f(x_3) > 0$, x_3 replaces x_2

Fourth iteration

$x_1 = 1.1111$
$x_2 = 1.1344$

$f(x_1) = - 0.0681$
$f(x_2) = 0.00077$

$$x_3 = 1.1344 - \frac{0.00077}{[0.0.00077 - (-0.0681)]}(1.1344 - 1.1111)$$
$$= 1.1344 - (0.01118)(0.0233)$$
$$= 1.1341$$

$$f(x_3) = 3(1.1341)^2 - (1.1341)^3 - 2.4$$
$$= 0.000111$$

Because $f(x_3) < 0$, x_3 replaces x_1

The root is therefore between 1.1341 and 1.1344.

c)
If the root of an equation is x^*, Then the actual error of the i^{th} iteration of a numerical method is:

$$E_i = x^* - x_i$$

The relationship between these successive terms can be expressed as follows:

$$E_{i+1} = k \times (E_i)^r$$

Where r is the convergence rate. For the iteration i, we write:

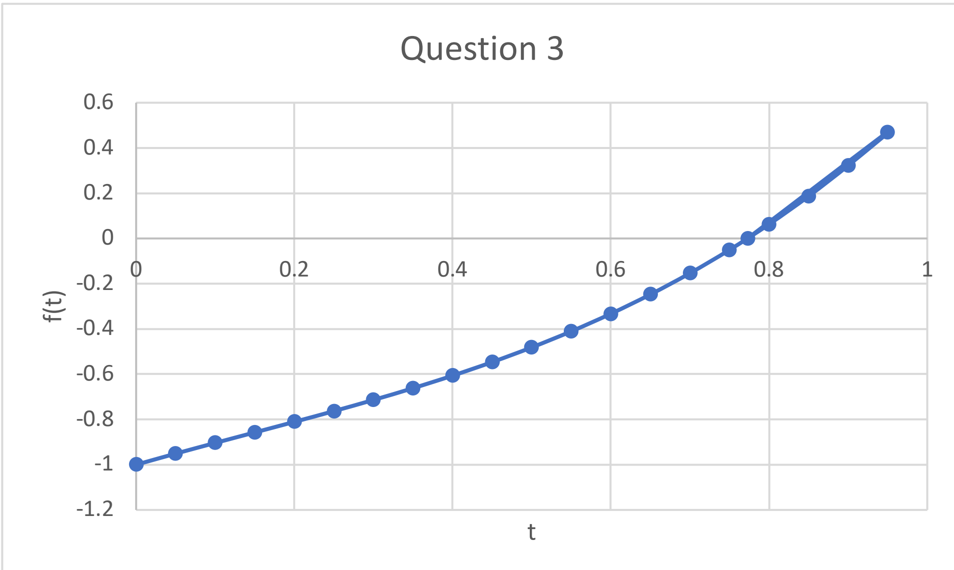
$$r = \frac{\ln(E_i) - \ln(E_{i-1})}{\ln(E_{i-1}) - \ln(E_{i-2})}$$

Iteration	***x****	***x_i = x_3***	***E_i***
2	1.13413785	1.1111	0.02304
3	1.13413785	1.1344	0.000262
4	1.13413785	1.1341	0.00003785

$$r = \frac{\ln(0.00003785) - \ln(0.000262)}{\ln(0.000262) - \ln(0.02304)}$$
$$= \frac{-1.9347135}{-4.4766426}$$
$$= 0.432$$

QUESTION 3
a)

$$\boldsymbol{f(x) = t^3 - e^{-t}}$$



b)
<u>First iteration</u>

$$x_3 = \frac{x_1 + x_2}{2} = \frac{0+1}{2} = 0.5$$
$$f(x_3) = 0.5^3 - e^{-0.5} = -0.48153$$

$$f(x_3) \cdot f(x_1) = (-0.48153)(-1) > 0$$

x_3 replaces x_1.
<u>Second iteration</u>

$$x_3 = \frac{x_1 + x_2}{2} = \frac{0.5+1}{2} = 0.75$$
$$f(x_3) = 0.75^3 - e^{-0.75} = -0.05049$$

$$f(x_3) \cdot f(x_1) = (-0.05049)(-1) > 0$$

x_3 replaces x_1.
<u>Third iteration</u>

$$x_3 = \frac{x_1 + x_2}{2} = \frac{0.75+1}{2} = 0.875$$
$$f(x_3) = 0.875^3 - e^{-0.875} = 0.25306$$

$$f(x_3) \cdot f(x_1) = (0.25036)(-0.05049) < 0$$

x_3 replaces x_2.
<u>Fourth iteration</u>

$$x_3 = \frac{x_1 + x_2}{2} = \frac{0.75+0.875}{2} = 0.8125$$
$$f(x_3) = 0.8125^3 - e^{-0.8125} = 0.09263$$

$$f(x_3) \cdot f(x_1) = (0.09263)(-0.05049) < 0$$

x_3 replaces x_2.

The root is therefore between [0.75. 0.8125].

The error of the last iteration:

$$100\% \frac{x_i - x_{i-1}}{x_i}$$

$$100\% \frac{0.8125 - 0.875}{0.8125} = -7.69\%$$

c)

First iteration

$$x_1 = 0; f(x_1) = -1$$
$$x_2 = 1; f(x_2) = 0.63212$$

$$x_3 = x_2 - \frac{f(x_2)}{f(x_2) - f(x_1)}(x_2 - x_1)$$
$$x_3 = 1 - \frac{0.63212}{0.63212 - (-1)}(1 - 0) = 0.6127$$
$$f(x_3) = 0.6127^3 - e^{-0.6127} = -0.31188$$

x_3 and x_1 have the same sign, thus x_1 is replaced by x_3.

Second iteration

$$x_1 = 0.6127; f(x_1) = -0.31188$$
$$x_2 = 1; f(x_2) = 0.63212$$

$$x_3 = 1 - \frac{0.63212}{0.63212 - (-0.31188)}(1 - 0.6127) = 0.74066$$
$$f(x_3) = 0.7466^3 - e^{-0.7466} = -0.0705$$

x_3 and x_1 have the same sign then x_1 is replaced by x_3.

Third iteration

$$x_1 = 0.74066; f(x_1) = -0.0705$$
$$x_2 = 1; f(x_2) = 0.63212$$

$$x_3 = 1 - \frac{0.63212}{0.63212 - (-0.0705)}(1 - 0.74066) = 0.76668$$
$$f(x_3) = 0.76668^3 - e^{-0.76668} = -0.0139$$

x_3 and x_1 have the same sign, thus x_1 is replaced by x_3.

Fourth iteration

$$x_1 = 0.76668; f(x_1) = -0.0139$$

$$x_2 = 1; f(x_2) = 0.63212$$

$$x_3 = 1 - \frac{0.63212}{0.63212 - (-0.0139)}(1 - 0.76668) = 0.7717$$
$$f(x_3) = 0.7717^3 - e^{-0.7717} = -2.663 \times 10^{-3}$$

The root is between [0.7717, 1].

The error of the last iteration:

$$100\% \frac{x_i - x_{i-1}}{x_i}$$
$$100\% \frac{0.7717 - 0.76668}{0.7717} = 0.65\%$$

d)

$$f(x_0) = f(0) = -1$$
$$f(x_1) = f(1) = 0.63212$$

$$x_2 = x_1 - \frac{f(x_1)}{f(x_0) - f(x_1)}(x_0 - x_1)$$

First iteration (see part c)

$$x_2 = 0.6127$$
$$f(x_0) = 0.63212$$
$$f(x_1) = -0.31188$$

Second iteration

$$x_2 = 0.6127 - \frac{-0.31188}{0.63212 - (-0.31188)}(1 - 0.6127) = 0.74066$$
$$f(x_0) = f(0.6127) = -0.31188$$
$$f(x_1) = f(0.74066) = -0.07049$$

Third iteration

$$x_2 = 0.74066 - \frac{-0.07049}{-0.31188 - (-0.07049)}(0.6127 - 0.74066) = 0.77803$$
$$f(x_0) = f(0.74066) = -0.07049$$
$$f(x_1) = f(0.77803) = 0.011654$$

Fourth iteration

$$x_2 = 0.77803 - \frac{0.011654}{-0.07049 - (0.011654)}(0.74066 - 0.77803) = 0.77273$$

$$f(x_1) = f(0.77273) = -3.487 \times 10^{-4}$$

The error of the last iteration:

$$100\% \frac{x_i - x_{i-1}}{x_i}$$

$$100\%\frac{0.77273-0.77803}{0.77273}=-0.686\%$$

e)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$f'(t) = 3t^2 + e^{-t}$$

<u>First iteration</u>

$$f(x_1) = f(0.1) = -0.903837$$
$$f'(x_1) = 0.934837$$
$$x_2 = 0.1 - \frac{-0.903837}{0.934837} = 1.066839$$

<u>Second iteration</u>

$$f(x_1) = f(1.066839) = 0.870123$$
$$f'(x_1) = 3.758531$$
$$x_2 = 0.835333$$

<u>Third iteration</u>

$$f(x_1) = f(0.835333) = 0.14915$$
$$f'(x_1) = 2.527074$$
$$x_2 = 0.776312$$

<u>Fourth iteration</u>

$$f(x_1) = f(0.776312) = 0.007753$$
$$f'(x_1) = 2.268082$$
$$x_2 = 0.772894$$

The root is approximated by 0.772894.

The error of the last iteration:

$$100\%\frac{x_i - x_{i-1}}{x_i}$$
$$100\%\frac{0.772894-0.776312}{0.772894}=-0.44\%$$

f)
First candidate

$$t = \sqrt[3]{e^{-t}}\ g'(x) = \frac{\sqrt[3]{e^{-t}}}{-3}$$

Second candidate

$$t = -3ln(t)\ \ g'(x) = -\frac{3}{t}$$

Third candidate

$$t = \frac{e^{-t}}{t^2}\ \ g'(x) = -\frac{e^{-t}(t+2)}{t^3}$$

Fourth candidate
A “+t « is added on each side of the equation to get:

$$t^3 + t = e^{-t} + t$$

$$t = e^{-t} - t^3 + t \quad g'(x) = -e^{-t} - 3t^2 + 1$$

The convergence criterion is $|g'(t)| < 1$ in the region that the root is searched. Because the iteration starts with $t = 0$, note that the first or the fourth candidate is valid.

Using the first candidate:

Iteration	t_0	t_1
1	0	1
2	1	0.716531
3	0.716531	0.787538
4	0.787538	0.769117

g)

Summary

Part	Root	Relative error
B	[0.75, 0.8125]	-7.69%
C	[0.7717, 1]	0.65%
D	0.77273	-0.69%
E	0.772894	-0.44%

The root of the function is approximately $x = 0.772882739147897$.
It can be concluded that Newton-Raphson's method seems to converge the fastest.