Assignment #5 – **NUMERICAL INTEGRATION AND DIFFERENTIATION**

Solve the following problems by hand. When applicable, verify your results via a spreadsheet.

Q.1 – Calculate the forward difference and the backward difference for O(h) and O(h^2) as well as the central difference for O(h^2) and O(h^4) for the first derivative of:

$$y = sin(x); x = \pi/4; h = \pi/12$$

Compare the actual relative error for each approximation.

Analytical value

$$f'\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = 0.7071$$

Forward difference

$$f'(x_i) = \frac{f(x_i + h) - f(x_i)}{h} \quad O(h), \qquad f'(x_i) = \frac{-f(x_i + 2h) + 4f(x_i + h) - 3f(x_i)}{2h} \quad O(h^2)$$

$$f'\left(\frac{\pi}{4}\right) = \frac{f\left(\frac{\pi}{4} + \frac{\pi}{12}\right) - f\left(\frac{\pi}{4}\right)}{\frac{\pi}{12}} = \frac{0.8660 - 0.7071}{0.2618} = 0.607$$

$$\epsilon = \frac{|0.7071 - 0.607|}{0.7071} = 14.1\%$$

$$f'\left(\frac{\pi}{4}\right) = \frac{-f\left(\frac{\pi}{4} + 2\frac{\pi}{12}\right) + 4f\left(\frac{\pi}{4} + \frac{\pi}{12}\right) - 3f\left(\frac{\pi}{4}\right)}{2\frac{\pi}{12}} = \frac{-0.9659 + 4 \cdot 0.8660 - 3 \cdot 0.7071}{0.5236}$$

$$= 0.7196$$

$$\epsilon = \frac{|0.7071 - 0.7196|}{0.7071} = 1.8\%$$

Backward difference

$$f'(x_i) = \frac{f(x_i) - f(x_i - h)}{h} \quad O(h), \qquad f'(x_i) = \frac{3f(x_i) - 4f(x_i - h) + f(x_i - 2h)}{2h} \quad O(h^2)$$

$$f'\left(\frac{\pi}{4}\right) = \frac{f\left(\frac{\pi}{4}\right) - f\left(\frac{\pi}{4} - \frac{\pi}{12}\right)}{\frac{\pi}{12}} = \frac{0.7071 - 0.5}{0.2618} = 0.7911$$

$$\epsilon = \frac{|0.7071 - 0.7911|}{0.7071} = 11.9\%$$

$$f'\left(\frac{\pi}{4}\right) = \frac{3f\left(\frac{\pi}{4}\right) - 4f\left(\frac{\pi}{4} - \frac{\pi}{12}\right) + f\left(\frac{\pi}{4} - 2\frac{\pi}{12}\right)}{2\frac{\pi}{12}} = \frac{3 \cdot 0.7071 - 4 \cdot 0.5 + 0.2588}{0.5236} = 0.7259$$

$$\epsilon = \frac{|0.7071 - 0.7259|}{0.7071} = 2.6\%$$

Central difference

$$(x_i) = \frac{f(x_i + h) - f(x_i - h)}{2h} O\left(h^2\right)$$

$$f'(x_i) = \frac{-f(x_i+2h)+8f(x_i+h)-8f(x_i-h)+f(x_i-2h)}{12h} \quad O(h^4)$$

$$f'\left(\frac{\pi}{4}\right) = \frac{f\left(\frac{\pi}{4}+\frac{\pi}{12}\right)-f\left(\frac{\pi}{4}-\frac{\pi}{12}\right)}{2\frac{\pi}{12}} = \frac{0.8660-0.5}{0.5236} = 0.6990$$

$$\epsilon = \frac{|0.7071-0.6990|}{0.7071} = 1.1\%$$

$$f'\left(\frac{\pi}{4}\right) = \frac{-f\left(\frac{\pi}{4}+2\frac{\pi}{12}\right)+8f\left(\frac{\pi}{4}+\frac{\pi}{12}\right)-8f\left(\frac{\pi}{4}-\frac{\pi}{12}\right)+f\left(\frac{\pi}{4}-2\frac{\pi}{12}\right)}{12\frac{\pi}{12}} =$$

$$\frac{-0.9659+8\cdot 0.8660-8\cdot 5+0.2588}{3.1416} = 0.7069$$

$$\epsilon = \frac{|0.7071-0.7069|}{0.7071} = 0.03\%$$

Summary

	O(h)		O(h²)		O(h⁴)	
	Estimated value	error (%)	Estimated value	error (%)	error (%)	error
Forward	0.6070	14.1	0.7196	1.8	-	-
Backward	0.7911	11.9	0.7259	2.6	-	-
Central	-	-	0.6990	1.1	0.7069	0.03

Q.2 – Use Richardson's interpolation to estimate the first derivative of

$$y = sin(x); x = \pi/4; h\_1 = \pi/3; h\_2 = \pi/6$$

Use the central difference O (h^2) for the initial estimate and compare your results with those obtained in question 1.

$$f'\left(\frac{\pi}{4}_{h=\frac{\pi}{3}}\right) = \frac{f\left(\frac{\pi}{4}+\frac{\pi}{3}\right)-f\left(\frac{\pi}{4}-\frac{\pi}{3}\right)}{2\left(\frac{\pi}{3}\right)} = \frac{0.9659+0.2588}{2.0943} = 0.5847$$

$$f'\left(\frac{\pi}{4}_{h=\frac{\pi}{6}}\right) = \frac{f\left(\frac{\pi}{4}+\frac{\pi}{6}\right)-f\left(\frac{\pi}{4}-\frac{\pi}{6}\right)}{2\left(\frac{\pi}{6}\right)} = \frac{0.9659-0.2588}{1.0471} = 0.6752$$

$$D \approx \frac{4}{3}D(h_2) - \frac{1}{3}D(h_1) = \frac{4}{3}(0.6752) - \frac{1}{3}(0.5847) = 0.7053$$

Note that accuracy is as same as central difference O(h^4).

Q.3 – Calculate the following integral with the trapezoidal method that will allow you to use Romberg with a precision O(h^8). In addition, estimate the integral with the Simpson 1/3 method with 4 intervals (thus 8 segments).

$$\int_0^2 \frac{e^x \cdot \sin(x)}{1+x^2} dx$$

Trapezoidal Rule:

$I_1 = \frac{2}{2}\ (f(0) + f(2)) = \ 1.34376994$

$I_2 = \frac{1}{2}\ \big(f(0) + 2f(1) + f(2)\big) = \ 0.5 * (0 + 2 * 1.14367764 + 1.34376994)$= 1.81556261

$I_3 = \frac{0.5}{2}\ \big(f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + f(2)\big) = \ 0.25 * (0 + 2 * 0.63235127 + 2 * 1.14367764 + 2 * 1.37552689 + 1.34376994)$= 1.91172038

$I_4 = \frac{0.25}{2}\ \big(f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + 2f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2)\big) = \ 0.25 * (0 + 2 * 0.298986332 + 2 * 0.63235127 + 2 * 0.92353873 + 2 * 1.14367764 + 2 * 1.292597772 + 2 * 1.37552689 + 2 * 1.39383339 + 1.34376994)$= 1.93309925

__**1/3 Simpson**__

h=0.25

$$\frac{h}{3}[f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + 2f(1) + 4f(1.25) + f(1.5) + 4f(1.75) + f(2)]$$

$$= \frac{0.25}{3}[0 + 4 \cdot 1.1959 + 2 \cdot 1.2647 + 4 \cdot 3.6942 + 2 \cdot 2.2874 + 4 \cdot 5.1704 + 2 \cdot 2.7511 + 4 \cdot 5.5753 + 1.3438] = 1.9402$$

Q.4 – Repeat question 3 but use the Romberg integration with precision O(h^8). Use the results obtained with the trapezoidal method in question 3. Present a summary of your results in a table.

Calculate the integral with the trapezoid rule with step sizes of 2, 1, 0.5 and 0.25 (done in Q3). Because we used the trapezoid rule, the index *k* is reduced to *1*.

$$I_{j,k} \cong \frac{\left[4^{k-1} \cdot I_{j,k-1}\right] - I_{j-1,k-1}}{4^{k-1} - 1}$$

We need now to find the values:

$$I_{j,2} = \frac{\left[4^1 \cdot I_{j,1}\right] - I_{j-1,1}}{4^1 - 1}$$

$$I_{2,2} = \frac{4 * I_{2,1} - I_{1,1}}{3} = 1.97282684$$

$$I_{3,2} = \frac{4 * I_{3,1} - I_{2,1}}{3} = 1.94377297$$

$$I_{4,2} = \frac{4 * I_{4,1} - I_{3,1}}{3} = 1.94022553$$

With k = 2, we can calculate the estimates of k=3 i.e. O(h^6).

$$I_{j,3} = \frac{\left[4^2 \cdot I_{j,2}\right] - I_{j-1,2}}{4^2 - 1}$$

$$I_{2,3} = \frac{4^2 * I_{2,2} - I_{1,2}}{4^2 - 1} = 1.94183605$$

$$I_{3,3} = \frac{4^2 * I_{3,2} - I_{2,2}}{4^2 - 1} = 1.93998904$$

Finally, with the estimated k = 3 we can evaluate k = 4 and thus O(h^8).

$$I_{j,4} = \frac{[4^3 \cdot I_{j,3}] - I_{j-1,3}}{4^3 - 1} = \frac{[64 \cdot I_{j,3}] - I_{j-1,3}}{63}$$

$$I_{4,4} = \frac{4^3 * I_{4,3} - I_{3,3}}{4^3 - 1} = 1.93995972$$

Summary

	j=1	j=2	j=3	j=4
No. intervals	1	2	4	8
h =	2	1	0.5	0.25
Trapezoidal Rule	1.34377	1.81556	1.91172	1.9331
Function evaluations	2	3	5	9
		1.97283	1.94377	1.94023
			1.94184	1.93999
				1.93996

Q.5 – Repeat question 3 but use Gaussian Quadrature. Use the Gauss-Legendre formula with six points (see reference on next page).

$$\int_0^2 \frac{e^b \sin(b)}{1+b^2}\,db = \int_0^2 \frac{e^{x+1}\sin(x+1)}{1+(x+1)^2}\,dx$$

Where x =Integral points
b= 1*x+1 and db=dx
c=weighting factors

x	b=x+1	f(x)	c	I(x)
-0.9324695	0.0675305	0.0718657	0.1713245	0.0123124
-0.6612094	0.3387906	0.4183478	0.3607616	0.1509238
-0.2386192	0.7613808	0.9351654	0.4679139	0.4375769
0.2386192	1.2386192	1.2872829	0.4679139	0.6023376
0.6612094	1.6612094	1.3948677	0.3607616	0.5032147
0.9324695	1.9324695	1.3644139	0.1713245	0.2337575
			Total	1.9401229

Example

$$f(-0.9324695) = \frac{e^{1*(-0.932469514)+1}\, sin(-0.932469514+1)}{1+(1*(-0.932469514)+1)^2} = 0.0718657$$

$$I_1 = f_1 * c_1 = 0.0718657 * 0.1713245 = 0.0123124$$

$I = \sum_{i=1}^{b}[c_{i\times} f(x_i)] = 1.9401229$ VI points (see reference on next page).

Q.6 – Compare the results obtained in questions 3, 4 and 5.

	Trapeze	**Simpson 1/3**	**Romberg**	**Gauss**	**Analytical value**
I	1.9331	1.9402	1.93995972	1.9401229	1.94013
ε (%)	0.36	0.0049	0.0088	0.00037	-

With h = 0.25 for trapeze and Simpson

$$\varepsilon\ (\%) = \left|\frac{(1.9331 - 1.94013)}{1.94013}\right| = 30.74\%$$

According to these results, we can conclude that the Gaussian method is the most accurate and that, as expected, the trapezium method is the least accurate. With enough intervals, Simpson 1/3 can be more accurate than Romberg. Usually for a number of operations equivalent it will be the opposite:

Gauss> Romberg> Simpsons 3/8> Simpson 1/3> Trapeze

Q.7 – The cross-sectional area of a stream of water can be estimated using

$$A_c = \int_0^B H(y)dy$$

The flow can be estimated with

$$Q = \int_0^B U(y)H(y)dy$$

Where: B is the width of the stream (m) and y is the distance from the bank (m).Using the Simpson 1/3 method and 1/3 or trapezoid method; evaluate the flow and cross-sectional area of the stream with the following properties:

y,m	0	2	4	5	6	9
H, m	0.5	1.3	1.25	1.7	1	0.25
U, m/s	0.03	0.06	0.05	0.12	0.11	0.02

Area

$$I_1 = (2-0)\frac{1.3+0.5}{2} = 1.8\, m^2$$
$$I_2 = (4-2)\frac{1.25+1.3}{2} = 2.55\, m^2$$
$$I_3 = (5-4)\frac{1.7+1.25}{2} = 1.475\, m^2$$
$$I_4 = (6-5)\frac{1+1.7}{2} = 1.35\, m^2$$
$$I_5 = (9-6)\frac{0.25+1}{2} = 1.875\, m^2$$
$$A_c = \sum I_i = 9.05\, m^2$$

Flow discharge

$$I_1 = (2-0)\frac{1.3\cdot 0.06+0.5\cdot 0.03}{2} = 0.093\, m^3/s$$
$$I_2 = (4-2)\frac{1.25\cdot 0.05+1.3\cdot 0.06}{2} = 0.1405\, m^3/s$$
$$I_3 = (5-4)\frac{1.7\cdot 0.12+1.25\cdot 0.05}{2} = 0.13325\, m^3/s$$
$$I_4 = (6-5)\frac{1\cdot 0.11+1.7\cdot 0.12}{2} = 0.157\, m^3/s$$

$$I_5 = (9-6)\frac{0.25 \cdot 0.02 + 1 \cdot 0.11}{2} = 0.1725\ m^3/s$$

$$Q = \sum I_i = 0.69625\ m^3/s$$

It is also possible to use Simpson for the interval 0-4 and the interval 4-6 and complete with the trapezium method. Watch out for irregular intervals!