Selected Exercises for Sections 5.4, 6.2, 6.7 and 6.8

1. Let b and c be real numbers. Prove that if the quadratic equation $x^2 + bx + c = 0$ has a nonreal complex solution z, then $\bar{z}$ is also a solution.

2. **Prove** or **disprove** the following statements. In each case, indicate first **which method** from the following six methods you are using:

   1) *Direct proof of implication*
   2) *Proof of the contrapositive of implication*
   3) *Indirect proof of implication by contradiction*
   4) *Constructive proof of existence*
   5) *Non-constructive proof of existence*
   6) *Disproof by counterexample*

   (a) Let b and c be nonzero **real** numbers. The statement is:

   If the quadratic equation $x^2 + bx + c = 0$ has a nonreal complex solution $z \neq 0$, then $w = \dfrac{1}{\bar{z}}$ is a solution of the equation $cx^2 + bx + 1 = 0$.

   The method: ---------

   (b) Let z and w be nonzero complex numbers. The statement is:

   If z is real and w is pure imaginary, then $|z + w| < |z| + |w|$.

   The method: ---------

   (c) Let z be a complex number. The statement is:

   If z^2 is a real number, then z is itself a real number.

   The method: ---------

   (d) Let z be a nonzero complex number. The statement is

   If $\bar{z} = z^2$, then $|z| = 1$.

   The method: ---------

   (e) The statement is:

   If $|w_1 + w_2| \leq |w_1| + |w_2|$ for arbitrary complex numbers w_1 and w_2, then $|z_1 - z_2| \geq |z_1| - |z_2|$ for arbitrary complex numbers z_1 and z_2.

   (f) Let z be a nonzero complex number. The statement is:

   Both z and w are nonreal if $|z + w| \neq |z| + |w|$.

   (g) Let ω be an n-th root of unity, where n is an even natural number. The statement is:

   If k is an even natural number, then $i^k \overline{\omega}^{2k-1}$ is also an n-th root of unity.

   (h) Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$. The statement is:

   If $a \not\equiv b \pmod{n}$, then $ac \not\equiv bc \pmod{n}$.

(i) Let $n \in \mathbb{N}$. Recall that the congruence class of an integer a modulo n is the subset $[a]_n = \{b \in \mathbb{Z} : b \equiv a \pmod{n}\}$ of the set of integers $\mathbb{Z}$. The statement is:

For any $n, m \in \mathbb{N} - \{1\}$, there exists an integer $k > 1$ such that $[0]_n \cap [0]_m \subseteq [0]_k$.

(j) Let m, n be consecutive positive odd numbers. The statement is:

There exist $x, y \in \mathbb{Z}$ such that $mx + ny = 1$.

3. For each problem in Question 2 above, provide an alternative proof with a different method if you have **proved** it.

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

can be used to give the conversion between standard form and exponential form of complex numbers.

Use complex numbers in exponential form instead of polar form in the rest of the problems.

4. (a) Convert each of the following complex numbers from standard form to exponential form:

   $w = -2 + 2i =$

   $z = \dfrac{1}{\sqrt{2}} - \dfrac{\sqrt{6}}{2}i =$

   (b) Compute

   $$\frac{w^4}{iz^5}$$

   with w and z in exponential form found in part (a).

   (c) Convert your answer in part (b) into a complex number in standard form.

5. (a) Find **all** complex numbers that satisfy the equation

   $$z^3 + 2 - 2i = 0,$$

   and express **all** solutions in **exponential** form.

   (b) Convert **any two** of your solutions in part (a) into complex numbers in **standard** form, in which all radicals should be in exact values.

6. (a) Convert each of the following complex numbers from standard form to exponential form:

   $u = 2 - 2i =$

   $w = -2 - 2\sqrt{3}i =$

   $z = -\dfrac{\sqrt{6}}{2} + \dfrac{1}{\sqrt{2}}i =$

(b) Compute

$$\frac{w^2 z^3}{u^4}$$

with u, w and z in exponential form found in part (a).

7. (a) Solve the equation

$$z^6 + 8i = 0,$$

and express **all** solutions in **exponential** (or polar) form.

(b) Convert any **three** of your solutions in part (a) into complex numbers in **standard** form, in which all radicals should be in exact values.

8. Prove that

$$\cos 3\theta = \cos\theta(4\cos^2\theta - 3)$$

by applying the **Binomial Theorem** and **De Moivre's Theorem** to $(\cos\theta + i\sin\theta)^3$.