Solution for Assignment on Column Buckling

Starting with the differential equation of equilibrium for column buckling $EIy^{iv} + Py'' = 0$, and its solution $y(x) = A\cos kx + B\sin kx + Cx + D$, in which $k = \sqrt{P/EI}$, it is required to formulate the buckling load P for a column fixed at both ends.

Solution:

Displacement and Derivative:

$$y = A\cos kx + B\sin kx + Cx + D$$

$$y' = -Ak\sin kx + Bk\cos kx + C$$

Boundary Conditions:

BC1: $at\ x = 0 \therefore y = 0 \Rightarrow y(0) = 0$

BC2: $at\ x = 0 \therefore y' = 0 \Rightarrow y'(0) = 0$

BC3: $at\ x = l \therefore y = 0 \Rightarrow y(l) = 0$

BC4: $at\ x = l \therefore y' = 0 \Rightarrow y'(0) = 0$

The above equations lead to the following four equations

$$A + D = 0$$

$$Bk + C = 0$$

$$A\cos kl + B\sin kl + Cl + D = 0$$

$$-Ak\sin kl + Bk\cos kl + C = 0$$

Determinant approach:

Express all the boundary conditions in a matrix form to obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & k & 1 & 0 \\ \cos kl & \sin kl & l & 1 \\ -k\sin kl & k\cos kl & 1 & 0 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Either vector $\langle A, B, C, D \rangle^T = \langle 0, 0, 0, 0 \rangle^T$ which leads to the trivial solution $y(x) = 0$ or the determinant must vanish.

As you may observe, the determinant leads to the same solution as the intuitive approach. While it is more systematic, it can be lengthier to implement.

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & k & 1 & 0 \\ \cos kl & \sin kl & l & 1 \\ -k\sin kl & k\cos kl & 1 & 0 \end{vmatrix} = 0$$

$$1\times\begin{vmatrix} k & 1 & 0 \\ \sin kl & l & 1 \\ k\cos kl & 1 & 0 \end{vmatrix} - 1\times\begin{vmatrix} 0 & k & 1 \\ \cos kl & \sin kl & l \\ -k\sin kl & k\cos kl & 1 \end{vmatrix} = 0$$

$$-1\times\begin{vmatrix} k & 1 \\ k\cos kl & 1 \end{vmatrix} + k\begin{vmatrix} \cos kl & l \\ -k\sin kl & 1 \end{vmatrix} - 1\times\begin{vmatrix} \cos kl & \sin kl \\ -k\sin kl & k\cos kl \end{vmatrix} = 0$$

$$-k + k\cos kl + k\cos kl + k^2 l\sin kl - k\cos^2 kl - k\sin^2 kl = 0$$

$$k\left(-2 + 2\cos kl + kl\sin kl\right) = 0$$

$$\left(-2 + 2\cos kl + kl\sin kl\right) = 0$$

$$-2\left(1 - \cos kl\right) + kl\sin kl = 0$$

$$-4\sin^2\frac{kl}{2} + 2kl\sin\frac{kl}{2}\cos\frac{kl}{2} = 0$$

$$-\sin^2\frac{kl}{2} + \frac{kl}{2}\sin\frac{kl}{2}\cos\frac{kl}{2} = 0$$

$$\sin\frac{kl}{2}\left[-\sin\frac{kl}{2} + \frac{kl}{2}\cos\frac{kl}{2}\right] = 0$$

$$\sin\frac{kl}{2} = 0 \Rightarrow \frac{kl}{2} = n\pi$$

or

$$-\sin\frac{kl}{2} + \frac{kl}{2}\cos\frac{kl}{2} = 0 \Rightarrow \tan\frac{kl}{2} = \frac{kl}{2} \Rightarrow \frac{kl}{2} = 4.49, \ldots.$$

The smallest positive root is $\frac{kl}{2} = \pi$

$$kl = 2\pi \Rightarrow \sqrt{\frac{P}{EI}}l = 2\pi \Rightarrow P = \frac{4\pi^2 EI}{l^2}$$