Model Solution for Assignment 4 of 4

Please ensure that you ***include your name and student number*** *on your submission.*

Your submission ***must be created using Microsoft Word, Google Docs, or LaTeX****.*

Your submission ***must be saved as a "pdf"*** *document and* ***have the filename "lastname_studentid_a4.pdf"*** *(using your last name and student number)*

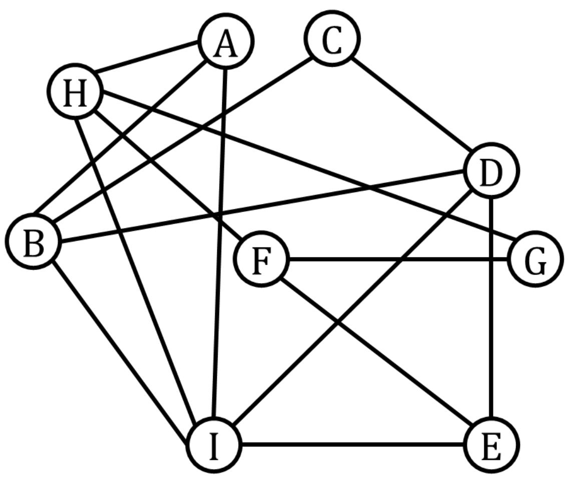
Do not compress your submission *into a "zip" file.*

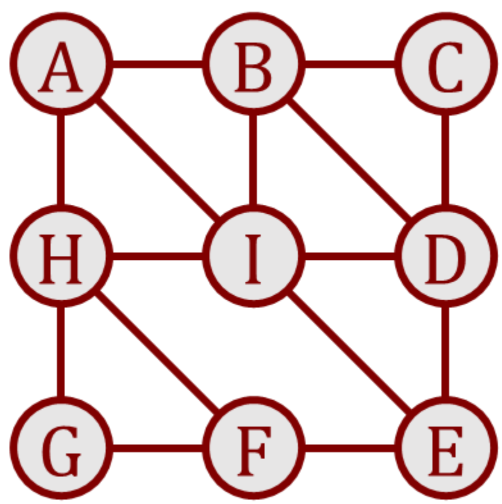
Late assignments will not be accepted *and will receive a mark of 0.*

Submissions ***written by hand****,* ***compressed into an archive****, or submitted in the* ***wrong format*** *(i.e., are not "pdf" documents)* ***will receive a mark of 0.***

The due date for this assignment is November 25, 2017, by 11:30pm.

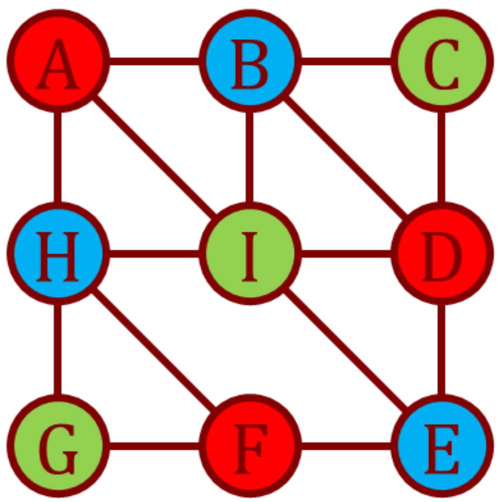
1. Draw a planar representation of the following graph. (4 marks)





Planar Representation
(One of Several Possibilities)

2. Provide a valid colouring for the graph in question 2 that uses the minimum number of colours. (Remember that a valid colouring is one that does not assign the same colour to two adjacent vertices.) Once you have found this colouring, prove that you have used the minimum number of colours. You are not permitted to reference any theorems for creating this proof - just find a valid colouring of n colours and then show that it is impossible to colour the graph in fewer than n colours. (4 marks)



To prove that you have used the minimum number of colours you must show that it is not possible to colour the graph in fewer colours. I have used three colours so I must prove that it cannot be coloured with only two.

Since I cannot reference any theorems, it would suffice for me to locate a subgraph that cannot be coloured in two colours. I could choose G' = ({B, C, D}, {{B, C}, {C, D}, {B, D}} and make the following proof by contradiction:

Proof (by Contradiction) that G' = (V' , E') cannot be coloured in 2 colours.

1. G' <u>can</u> be coloured in 2 colours
   let x be the colour of B
2. since {B, C} $\in$ E' and B is coloured x, C cannot be coloured x
   let y ($y \neq x$) be the colour of C
3. since {B, D} $\in$ E' and B is coloured x, D cannot be coloured x
   let z ($z \neq x$) be the colour of D
4. since {C, D} $\in$ E' and C is coloured y and D is coloured z, $y \neq z$
5. the minimum set of colours for G' is {x, y, z} $\wedge$ G' <u>can</u> be coloured in 2 colours
6. False

3. Identify the following sequences as either arithmetic or geometric. Specify the initial element and the common difference (if the sequence is arithmetic) or common ratio (if the sequence is geometric). (1 marks each)

   a. 4, 8, 16, 32, 64

      Geometric, a = 4, r = 2

   b. 2.25, 2.5, 2.75, 3, 3.25

      Arithmetic, a = 2.25, d = 0.25

   c. 1, 0.1, 0.01, 0.001, 0.0001

      Geometric, a = 1, r = 0.1

   d. 0.3, 0.09, 0.027, 0.0081, 0.00243

      Geometric, a = 0.3, r = 0.3

4. Prove that the sequence represented by the pattern $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$ is neither arithmetic nor geometric. (4 marks)

Proof by Contradiction that the sequence is Not Arithmetic:

1.	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$ is an Arithmetic Sequence	by Assumption
2.	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$ has form $a, a+d, a+2d, \ldots$	by Def^n of Arithmetic Sequence, 1
3.	$a = 1$	2
4.	$(a+d) = \frac{1}{2}$	2
5.	$(a+2d) = \frac{1}{3}$	2
6.	$a - (a+d) = 1 - \frac{1}{2}$	3, 4
7.	$d = -\frac{1}{2}$	6
8.	$(a+d) - (a+2d) = \frac{1}{2} - \frac{1}{3}$	4, 5
9.	$d = -\frac{1}{6}$	8
10.	$-\frac{1}{2} = -\frac{1}{6}$	7, 9
11.	False	

Proof by Contradiction that the sequence is Not Geometric:

1.	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$ is an Arithmetic Sequence	by Assumption
2.	$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$ has form $a, ar, ar^2, \ldots$	by Def^n of Geometric Sequence, 1
3.	$a = 1$	2
4.	$ar = \frac{1}{2}$	2
5.	$ar^2 = \frac{1}{3}$	2
6.	$\frac{ar}{a} = \frac{1}{2}$	3, 4/
7.	$r = \frac{1}{2}$	6
8.	$\frac{ar^2}{ar} = \frac{\frac{1}{3}}{\frac{1}{2}}$	4, 5
9.	$r = \frac{2}{3}$	8
10.	$\frac{1}{2} = \frac{2}{3}$	7, 9
11.	False	

5. What is the sum of all integers between 1 and 299 that are not divisible by either the number 7 or the number 9? You must use Sigma notation and the rules discussed in class to solve this problem and you must show all your work. (3 marks)

One important thing to note with this question is that there are some numbers that are divisible by both 7 and 9. You must ensure not to remove each of those numbers from the sum more than once (or, if you do remove them twice, you must add them back). To clarify, you are trying to compute the following:

{1, ..., 299} - {7, 14, 21, 28, 35, 42, 49, 56, 63, ...} - {9, 18, 27, 36, 45, 54, 63, ...} + {63, 126, ...}

n.b., I add back in 63 and 126 because they will both be deducted from the total *twice*!

$$\sum_{i=1}^{299} i - numbers\ divisible\ by\ 7\ or\ 9$$

$$= \sum_{i=1}^{299} i - \sum_{j=1}^{42} 7j - \sum_{k=1}^{33} 9k + \sum_{h=1}^{4} (7 \cdot 9)h$$

$$= \sum_{i=1}^{299} i - 7\sum_{j=1}^{42} j - 9\sum_{k=1}^{33} k + 63\sum_{h=1}^{4} h$$

$$= \frac{299(300)}{2} - 7\frac{42(43)}{2} - 9\frac{33(34)}{2} + 63\frac{4(5)}{2}$$

$$= 44850 - 7(903) - 9(561) + 63(10)$$

$$= 44850 - 6321 - 5049 + 630$$

$$= 34110$$

It's relatively easy to confirm this result experimentally too - using either a spreadsheet program like Microsoft Excel or the programming language of your choice!

6. Consider the following function written in Python 3 (recalling that range(x, y), in Python, refers to the sequence of values starting at x, counting up by 1s, and stopping at y-1.

```
for a in range(1, n+1):
     for b in range(1, a+1):
          foo()
          foo()
     for c in range(5, a+1):
          for d in range(1, c):
               bar()
```

If n has a value of 1000, how many times will the function `foo()` be called and how many times will the function `bar()` be called? You must solve this problem using Sigma notation (i.e., you must derive an expression that uses Sigma notation to specify how many times each of these functions will be called, and then you must find a closed form for this expression and evaluate for n = 1000). You must show all your work.

(6 marks)

The foo function is called twice inside the innermost of two nested loops. We would count this using Sigma notation as:

$$\sum_{a=1}^{n}\left(\sum_{b=1}^{a} 2\right)$$

$$= \sum_{a=1}^{n}(2a)$$

$$= 2\sum_{a=1}^{n} a$$

$$= 2\frac{n(n+1)}{2}$$

$$= 2\frac{n(n+1)}{2}$$

$$= 1001000$$

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The bar function is called once inside the innermost of three nested loops, but we have to watch the relationship between the outermost and the middle loop because when a is less than 5, the middle and innermost loops won't run at all! We would count this using Sigma notation as:

$$\sum_{a=1}^{n}\left(\sum_{c=5}^{a}\left(\sum_{d=1}^{c-1} 1\right)\right)$$

$$=\sum_{a=1}^{n}\left(\sum_{c=5}^{a}(c-1)\right)$$

It's important to recognize that the outer loop cannot start at 1 (or, more accurately, that when it does the inner loop is supposed to start 5 and stop at some value before 5, so the inner loop is never considered until a has a value greater than or equal to 5

$$=\sum_{a=5}^{n}\left(\sum_{c=5}^{a}(c-1)\right)$$

$$=\sum_{a=5}^{n}\left(\sum_{c=1}^{a-4}(c+3)\right)$$

$$=\sum_{a=5}^{n}\left(\sum_{c=1}^{a-4}c+\sum_{c=1}^{a-4}3\right)$$

$$=\sum_{a=5}^{n}\left(\sum_{c=1}^{a-4}c+3\sum_{c=1}^{a-4}1\right)$$

$$=\sum_{a=5}^{n}\left(\sum_{c=1}^{a-4}c+3(a-4)\right)$$

$$=\sum_{a=5}^{n}\sum_{c=1}^{a-4}c+\sum_{a=5}^{n}3(a-4)$$

$$=\sum_{a=5}^{n}\sum_{c=1}^{a-4}c+\sum_{a=1}^{n-4}3a$$

$$= \sum_{a=5}^{n} \sum_{c=1}^{a-4} c + 3\frac{(n-4)(n-3)}{2}$$

$$= \sum_{a=5}^{n} \frac{(a-4)(a-3)}{2} + 3\frac{(n-4)(n-3)}{2}$$

$$= \sum_{a=1}^{n-4} \frac{(a+4-4)(a+4-3)}{2} + 3\frac{(n-4)(n-3)}{2}$$

$$= \sum_{a=1}^{n-4} \frac{(a)(a+1)}{2} + 3\frac{(n-4)(n-3)}{2}$$

$$= \frac{1}{2}\left(\sum_{a=1}^{n-4} a^2 + a\right) + 3\frac{(n-4)(n-3)}{2}$$

$$= \frac{1}{2}\left(\sum_{a=1}^{n-4} a^2 + a\right) + 3\frac{(n-4)(n-3)}{2}$$

$$= \frac{1}{2}\left(\sum_{a=1}^{n-4} a^2 + \sum_{a=1}^{n-4} a\right) + 3\frac{(n-4)(n-3)}{2}$$

$$= \frac{1}{2}\left(\frac{(n-4)\big((n-4)+1\big)(2(n-4)+1)}{6} + \frac{(n-4)((n-4)+1)}{2}\right) + 3\frac{(n-4)(n-3)}{2}$$

$$= \frac{1}{2}\left(\frac{(n-4)\big((n-4)+1\big)(2(n-4)+1)}{6} + \frac{(n-4)((n-4)+1)}{2}\right) + 3\frac{(n-4)(n-3)}{2}$$

$$= \frac{1}{2}(329845486 + 496506) + 1489518$$

$$= 166660514$$

7. Let $f(x) = \frac{x^5}{99}$ and $g(x) = 2x^3$ and use a formal proof by contradiction to show that $f(x)$ is not $O(g(x))$. Please note the following: (4 marks)

$$
\begin{aligned}
0.010101010 &= f(1) < g(1) = 2\\
0.323232323 &= f(2) < g(2) = 16\\
2.454545455 &= f(3) < g(3) = 54\\
10.34343434 &= f(4) < g(4) = 128
\end{aligned}
$$

...

Proof by Contradiction

1. $f(x)$ is $O(g(x))$ by Assumption
2. $\exists c \left(c > 0 \land \exists k \left(k > 0 \land \forall x \left(x \geq k \rightarrow f(x) \leq cg(x)\right)\right)\right)$ by Definition, 1
3. $c' > 0 \land \exists k \left(k > 0 \land \forall x \left(x \geq k \rightarrow f(x) \leq c'g(x)\right)\right)$ by Existential Instantiation, 2
4. $\exists k \left(k > 0 \land \forall x \left(x \geq k \rightarrow f(x) \leq c'g(x)\right)\right)$ by Simplification, 3
5. $k' > 0 \land \forall x \left(x \geq k' \rightarrow f(x) \leq c'g(x)\right)$ by Existential Instantiation, 4
6. $\forall x \left(x \geq k' \rightarrow f(x) \leq c'g(x)\right)$ by Simplification, 5
7. $\forall x \left(x \geq k' \rightarrow \frac{x^5}{99} \leq c'(2x^3)\right)$
8. $\forall x \left(x \geq k' \rightarrow x^5 \leq 198c'x^3\right)$
9. $\forall x \left(x \geq k' \rightarrow x^2 \leq 198c'\right)$
10. $\forall x \left(x \geq k' \rightarrow x \leq \sqrt{198c'}\right)$

Strictly speaking there are two cases to handle here, but you will notice that the steps are identical for each case...

Case 1: $k' \geq \sqrt{198c'}$

11. $\forall x \left(x \geq k' \rightarrow x \leq \sqrt{198c'}\right)$ by Universal Instantiation (of x to k')
12. $k' \geq k' \rightarrow k' \leq \sqrt{198c'}$
13. $True \rightarrow False$
14. $False$

Case 2: $k' < \sqrt{198c'}$

11. $\forall x \left(x \geq k' \rightarrow x \leq \sqrt{198c'}\right)$ by Universal Instantiation (of x to $\sqrt{198c'}$+1)
12. $\sqrt{198c'} + 1 \geq k' \rightarrow \sqrt{198c'} + 1 \leq \sqrt{198c'}$
13. $True \rightarrow False$
14. $False$

8. Determine whether or not the following are true and provide a full derivation explaining your answer for each. The domain of the functions of n below is the positive real numbers. For convenience, you may assume that the logs are in the base of your choice, but you should specify what base you are using in your derivation.

(2 marks each)

a. $\frac{1}{n} + 15 - 3$ is $O(n^3)$

$\frac{1}{n} + 15 - 3$

$\leq \frac{1}{n} + 15$

$\leq \frac{1}{n} + 15n^3$

when $n \geq 1$

$\leq n^3 + 15n^3$

$\leq 16n^3$

a full proof (which was not required for this question) would continue as follows

$\forall n \in \mathbb{Z} \left(n \geq 1 \rightarrow \frac{1}{n} + 15 - 3 \leq 16n^3\right)$

$\exists k \left(\forall n \in \mathbb{Z} \left(n \geq k \rightarrow \frac{1}{n} + 15 - 3 \leq 16n^3\right)\right)$

$\exists c \left(\exists k \left(\forall n \in \mathbb{Z} \left(n \geq k \rightarrow \frac{1}{n} + 15 - 3 \leq cn^3\right)\right)\right)$

b. $3n \log n$ is $O(n^2)$

$3n \log n$

$\leq 3n^2$

when $n \geq 1$

Q.E.D. with c = 3, k = 1

c. $9n^2 + 3n^3 + n \log n$ is $\Omega(n^3)$

$9n^2 + 3n^3 + n \log n$

$\geq 3n^3 + n \log n$

when $n \geq 1$

$\geq 3n^3$

Q.E.D. with c = 3, k = 1

d. $(5n-4)^3$ is $\Theta(n^3)$

$(5n-4)^3 = (25n^2 - 40n + 16)(5n-4) = 125n^3 - 300n^2 + 240n - 64$
$125n^3 - 300n^2 + 240n - 64$
$\leq 125n^3 + 240n$
$\leq 125n^3 + 240n^3$
when $n \geq 1$
$\leq 365n^3$
$(5n-4)^3$ is $O(n^3)$ with c = 365, k = 1

$(5n-4)^3$
$\geq (5n-n)^3$ when $n \geq 4$
$\geq 4n^3$ when $n \geq 4$
$(5n-4)^3$ is $\Omega(n^3)$ with c = 4, k = 4

$(5n-4)^3$ is $O(n^3) \wedge (5n-4)^3$ is $\Omega(n^3) \leftrightarrow (5n-4)^3$ is $\Omega(n^3)$

e. $7n^2 - 3 + 2n$ is $O(n^2)$

$7n^2 - 3 + 2n$
$\leq 7n^2 + 2n$
$\leq 7n^2 + 2n^2$
when $n \geq 1$
$\leq 9n^2$
Q.E.D. with c = 9, k = 1

f. $\frac{6\log(n+2)}{3}$ is $O(n)$

$\frac{6\log(n+2)}{3} = 2\log(n+2)$
$2\log(n+2)$
$\leq 2(\text{n}+2)$
when $n \geq 1$
$\leq 2\text{n} + 4$
$\leq 2\text{n} + 4\text{n}$
$\leq 6n$
Q.E.D. with c = 6, k = 1

g. $\log(\frac{1}{n^4} + n^3)$ is $O(n^2)$

$\log(\frac{1}{n^4} + n^3)$
$\leq \log(n^4 + n^3)$
when $n \geq 1$
$\leq \log(n^4 + n^4)$
$\leq \log(2n^4)$
$\leq \log(n^5)$
when $n \geq 2$
$\leq 5 \log n$
$\leq 5n^2$
Q.E.D. with c = 5, k = 2

h. $9n^2$ is $\Omega(n^2)$

$9n^2$
$\geq 9n^2$
Q.E.D. with c = 9, k = 1