

SIMON FRASER UNIVERSITY
DEPARTMENT OF MATHEMATICS

Midterm 2

MATH 232 D100 Fall 2015

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November 9, 2015, 11:30 – 12:20 p.m.

Name: SOLUTIONS
family name *given name*

SFU ID: _____
student number *SFU-email*

Signature: _____

Instructions:

1. Do not open this booklet until told to do so.
2. Write your name above in block letters. Write your SFU student number and email ID on the line provided for it.
3. Write your answer in the space provided below the question. If additional space is needed then use the back of the previous page. Your final answer should be simplified as far as is reasonable.
4. Make the method you are using clear in every case unless it is explicitly stated that no explanation is needed.
5. This exam has 3 questions on 5 pages (not including this cover page). Once the exam begins please check to make sure your exam is complete.
6. **No** calculators, books, papers, or electronic devices shall be within the reach of a student during the examination.
7. During the examination, communicating with, or deliberately exposing written papers to the view of, other examinees is forbidden.

Question	Maximum	Score
1	12	
2	12	
3	12	
Total	36	

1.

- [1] (a) Define an elementary matrix.

Elementary matrix is a matrix which results from an identity matrix by applying a single elementary row operation to it.

- [1] (b) Answer TRUE or FALSE: An elementary matrix is invertible and the inverse is also an elementary matrix. (Do not need to justify.)

TRUE

- [3] (c) Show that if
- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{bmatrix}$
- is an elementary matrix, then
- a
- or
- b
- must be 0.

Suppose $a \neq 0$ and $b \neq 0$. To obtain an identity matrix from A , two of the three (non-zero) elements ^{in last row} must be changed to 0, using only one elementary operation. It follows that this operation cannot be "exchanging rows" or "multiplying row by a scalar". Therefore the identity matrix must result by "adding a multiple of row 1 or row 2 to row 3". However row 1 and row 2 both have only one 1, so this also cannot produce I_3 , a contradiction.

- [3] (d) Compute the inverse of A by the Inversion Algorithm. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] +(-1)R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array} \right] (-1)$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array} \right] +(-1)R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array} \right] \text{ DONE}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array} \right] +R_3$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

- [2] (e) Given nonempty set of vectors $S = \{v_1, v_2, \dots, v_s\}$ in \mathbb{R}^n . Define when these vectors are linearly independent.

Vectors in S are linearly independent if the only scalars that satisfy equation

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_s \vec{v}_s = \vec{0}$$

are $c_1 = c_2 = \dots = c_s = 0$.

- [2] (f) Determine whether the vectors are linearly independent: $(2, 4, 6, 2)$, $(5, 7, -2, -1)$, $(1, 1, 1, 1)$, $(-2, 1, -4, 3)$, $(2, 4, 3, -7)$. Justify your answer.

These vectors are from \mathbb{R}^4 . We know that a set of more than n vectors in \mathbb{R}^n is linearly dependent. As we are given five vectors in \mathbb{R}^4 , they are linearly dependent.

2.

- [2] (a) Define the column space of a matrix
- A
- .

The column space of $A_{m \times n}$ is the subspace of \mathbb{R}^m spanned by the column vectors of A .

$$\text{col}(A) = \text{span} \{ \vec{c}_1(A), \vec{c}_2(A), \dots, \vec{c}_n(A) \}$$

- [1] (b) Answer TRUE or FALSE: A linear system
- $Ax = b$
- is consistent if and only if
- b
- is not in the column space of
- A
- . (Do not need to justify.)

FALSE

- [3] (c) Find the parametric equation for the orthogonal complement
- a^\perp
- of
- $a = (4, 0, -7)$
- in
- \mathbb{R}^3
- .

$$a^\perp = \{ (x, y, z) : a \cdot (x, y, z) = 0 \} \text{ Hence}$$

$$a^\perp = \left\{ \left(\frac{4}{7}t, s, \frac{4}{7}t \right) : t, s \in \mathbb{R} \right\} \text{ and } \begin{cases} x = t \\ y = s \\ z = \frac{4}{7}t \end{cases} \quad t, s \in \mathbb{R}$$

- [3] (d) Given vectors
- $v_1 = (5, 4, 2)$
- ,
- $v_2 = (-1, 2, 3)$
- , and
- $v_3 = (6, 2, -1)$
- . Set up a linear system (you do not need to solve it) that can be used to test whether the vector
- $w = (3, 2, 0)$
- can be expressed as a linear combination of
- v_1, v_2
- , and
- v_3
- . State the condition on the system that guarantees there is such a linear combination.

$$\begin{bmatrix} 5 & -1 & 6 \\ 4 & 2 & 2 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

Vector \vec{w} is a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ if and only if the above system is consistent.

- [3] (e) Define a symmetric matrix. Then prove that a product of two symmetric matrices of the same size is a symmetric matrix if and only if the two matrices commute.

Definition: A square matrix is symmetric if $A^T = A$.

Proof: $(AB)^T = B^T A^T = BA$. Hence,

$$(AB)^T = AB \Leftrightarrow AB = BA.$$

Finally $AB = BA$ means the two matrices commute.

3.

- [1] (a) Answer TRUE or FALSE: If A is a triangular matrix, then the determinant $\det(A)$ is the sum of the entries on the main diagonal. (Do not need to justify.)

FALSE

- [2] (b) Let $A = (a_{ij})$ be an $n \times n$ square matrix. Define the cofactor expansion along the i -th row of A using minors M_{ij} of A . What is this expansion equal to?

cofactor expansion along the i -th row of A :

$$\sum_{j=1}^n a_{ij} C_{ij} = \det(A), \text{ where } C_{ij} = (-1)^{i+j} M_{ij} \text{ for}$$

$$i=1..n, j=1..n.$$

- [3] (c) Evaluate the determinant by the cofactor expansion along the first column.

$$\begin{vmatrix} \downarrow & & \\ 1 & 2 & -1 \\ -2 & 1 & 3 \\ 4 & -1 & 2 \end{vmatrix} = 1 \cdot (-1)^2 \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} + (-2)(-1)^3 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 4(-1)^4 \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} =$$

$$= 2 + 3 + 2(4 - 1) + 4(6 + 1) = 5 + 6 + 28 = \underline{\underline{39}}$$

- [3] (d) Let B be a square matrix obtained from A by multiplying a single row of A by a scalar k .

$$\text{Then } \det(B) = k \cdot \det(A)$$

Let B be a square matrix obtained from A by interchanging two rows of A .

$$\text{Then } \det(B) = -\det(A)$$

Let B be a square matrix obtained from A by adding a non-zero multiple of one row of A to another.

$$\text{Then } \det(B) = \det(A)$$

- [3] (e) Solve the linear system using Cramer's rule.

$$4x + 2y = -8$$

$$11x + y = 2$$

$$A = \begin{bmatrix} 4 & 2 \\ 11 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -8 & 2 \\ 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & -8 \\ 11 & 2 \end{bmatrix}$$

$$\det(A) = 4 - 22 = -18 \neq 0, \quad \det(A_1) = -8 - 4 = -12, \quad \det(A_2) = 8 + 88 = 96$$

$$x = \frac{-12}{-18} = \underline{\underline{\frac{2}{3}}}, \quad y = \frac{96}{-18} = \underline{\underline{-\frac{16}{3}}}$$