Name: Solutions

Lab Section: (circle your lab section)

room/time	9:00AM	1:30PM
LH3060	L1	L2
LH3066	L3	L4

*LH3060 is the computer lab closest to the elevators

WILFRID LAURIER UNIVERSITY
Waterloo, Ontario
Mathematics 104 – Calculus II
Midterm – February 27, 2018, 5:30PM

Instructor:
Dr. Chester Weatherby

Time Allowed: *85 minutes*

Total Value: *60 marks*

Number of Pages: *8 plus 2-sided cover page*

Instructions:

The CASIO FX-300MS Plus calculator is permitted. No other aids are allowed.

Check that your test paper has no missing, blank, or illegible pages.

Answer in the spaces provided. ***Please note that questions are printed on both sides of the test pages.*** *If you require more space on any question, make a note and continue on page 7.*

Show all your work. Insufficient justification will result in a loss of marks. Partial credit will be awarded according to the completeness of your work.

The formulae on page 8 are provided for your convenience.

Student Number: ______________

For grading purposes (leave blank)

Q #	Score	Out of		Q #	Score	Out of
1		6		6		5
2		7		7		7
3		7		8		9
4		3		9		9
5		7				

Total: out of 60 (possible 63)

[6 marks] 1. Evaluate $\int \sin^3(x)\cos^2(x)\,dx$.

$= \int \sin^2(x)\cos^2(x)\sin(x)\,dx$ sub $\sin^2 x = 1-\cos^2 x$

$= \int (1-\cos^2 x)\cos^2 x \sin x\,dx$ let $u = \cos x$, $du = -\sin x\,dx$

$= -\int (1-u^2)u^2\,du$

$= -\int u^2 - u^4\,du = -\frac{u^3}{3} + \frac{u^5}{5} + C$

$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C$

[5 marks] 2. a) Evaluate the improper integral $\int_1^\infty xe^{-x}\,dx$ or show that it is divergent.

int. by parts $uv - \int v\,du$

$u = x$ $dv = e^{-x}dx$

$du = dx$ $v = -e^{-x}$

$= \lim_{t\to\infty} \int_1^t xe^{-x}$

$= \lim_{t\to\infty} -xe^{-x}\Big]_1^t - \int_1^t -e^{-x}\,dt$

$= \lim_{t\to\infty} -xe^{-x} - e^{-x}\Big]_1^t$

$= \lim_{t\to\infty} \left(\frac{-t}{e^t} - \underbrace{e^{-t}}_{0}\right) - \left(-(1)e^{-1} - e^{-1}\right) = 2e^{-1}$

$= \frac{2}{e}$ convergent.

$\frac{\infty}{\infty}$

L'Hôp $\lim_{t\to\infty} \frac{-1}{e^t} = 0$

[2 marks] b) By comparison to the integral in part a) on the interval $(1,\infty)$, determine whether the improper integral $\int_0^\infty e^{-x}\,dx$ is convergent or divergent. DO NOT EVALUATE THE INTEGRAL.

On $(1,\infty)$ $1 < x$ so $e^{-x} < xe^{-x}$ (e^{-x} is pos.).

By comparison, $0 < \int_1^\infty e^{-x}dx < \int_1^\infty xe^{-x}$, so $\int_1^\infty e^{-x}dx$

converges (to a finite number), by part (a).

Now, $\int_0^\infty e^{-x}dx = \underbrace{\int_0^1 e^{-x}dx}_{\text{finite a area.}} + \underbrace{\int_1^\infty e^{-x}dx}_{\text{finite/conv.}}$

Thus $\int_0^\infty e^{-x}dx$ converges.

Over

[7 marks] **3.** Use an appropriate **trigonometric substitution** to evaluate $\int \frac{x}{\sqrt{4+x^2}}\,dx$.

$x = 2\tan\theta$
$dx = 2\sec^2\theta\, d\theta$

$= \int \frac{2\tan\theta}{\sqrt{4+4\tan^2\theta}} \, 2\sec^2\theta\, d\theta$

ID: $1+\tan^2\theta = \sec^2\theta$

$= 4\int \frac{\tan\theta\sec^2\theta}{2\sqrt{1+\tan^2\theta}}\, d\theta$

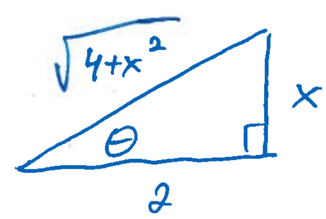
$= 2\int \frac{\tan\theta\sec^2\theta}{\sqrt{\sec^2\theta}}\, d\theta = 2\int \frac{\tan\theta\sec^2\theta}{\sec\theta}\, d\theta$ (ok to drop abs. value)

$= 2\int \sec\theta\tan\theta\, d\theta = 2\sec\theta + C$

ref Δ for
$x = 2\tan\theta$
$\tan\theta = \frac{x}{2}$ – opp / – adj

$\cos\theta = \frac{2}{\sqrt{4+x^2}}$

$\sec\theta = \frac{\sqrt{4+x^2}}{2}$



$= 2\left(\frac{\sqrt{4+x^2}}{2}\right) + C$

$= \sqrt{4+x^2} + C$

[3 marks] **4.** a) Use **Simpson's Rule** with $n = 4$ to approximate $\int_0^4 (3-x)^3\,dx$.

$\Delta x = \frac{4-0}{4} = 1$

$\int_0^4 (3-x)^3 dx \approx \frac{\Delta x}{3}\left(f(0)+4f(1)+2f(2)+4f(3)+f(4)\right)$

$= \frac{1}{3}\left(3^3+4(2)^3+2(1)^3+4(0)^3+(-1)^3\right)$

$= \frac{1}{3}\left(27+32+2+0-1\right) = 20$

[3 marks] b) **BONUS:** State the error bound formula for Simpson's Rule and use the formula to prove that the approximation in part a) is in fact the exact value of the integral.

$a=0$
$b=4$
$n=4$

$|E_S| \le \frac{K(b-a)^5}{180n^4}$ if $|f^{(4)}(x)| \le K$

$f(x) = (3-x)^3$
$f' = -3(3-x)^2$
$f'' = 6(3-x)$
$f''' = -6$
$f^{(4)} = 0$ (always).

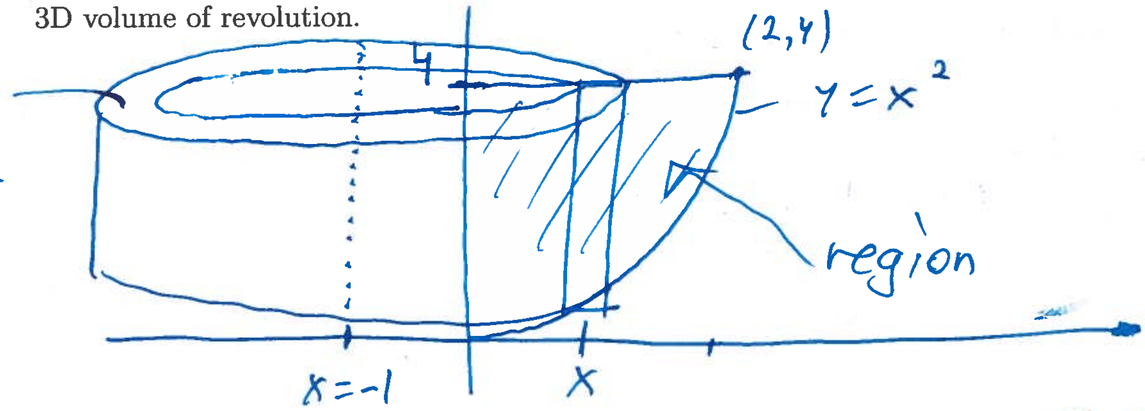
$K = 0$ is an upper bound for $f^{(4)}$, so $0 \le |E_S| \le \frac{0(4-0)^5}{180(4)^4}$

and so $E_S = 0$, Thus 0 error & Simpson's rule is the exact value of $\int_0^4 (3-x)^3 dx$.

Over

5. The region in the first quadrant bounded by $y = x^2$ and $y = 4$ is to be rotated about the line $x = -1$.

[3 marks] a) Sketch the 2D region to be rotated, as well as a single example of a cylindrical shell for the 3D volume of revolution.



[4 marks] b) Find the volume of revolution by using the method of cylindrical shells. $0 \le x \le 2$

$$V = \int_0^2 2\pi(x+1)(4-x^2)\,dx$$

$$= 2\pi \int_0^2 4x + 4 - x^3 - x^2\,dx$$

$$= 2\pi\left(2x^2 + 4x - \frac{x^4}{4} - \frac{x^3}{3}\right)\Big]_0^2$$

$$= 2\pi\left(8 + 8 - \frac{16}{4} - \frac{8}{3}\right)$$

$$\frac{36}{3} - \frac{8}{3} = \frac{28}{3}$$

$$= 2\pi\left(12 - \frac{8}{3}\right)$$

$$= 2\pi\left(\frac{28}{3}\right) = \frac{56\pi}{3}$$

[5 marks] 6. Find the exact area of the surface obtained by rotating the curve $y = \sqrt{1+2x}$ for $1 \le x \le 7$ about the x-axis.

int. wrt x

$f' = \frac{\cancel{2}}{\cancel{2}\sqrt{1+2x}}$

$$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$= \int_1^7 2\pi\sqrt{1+2x}\sqrt{1+\frac{1}{1+2x}}\,dx$$

$$= \int_1^7 2\pi\cancel{\sqrt{1+2x}}\,\frac{\sqrt{1+2x+1}}{\cancel{\sqrt{1+2x}}}\,dx$$

$$= 2\pi\int_1^7 (2+2x)^{1/2}\,dx$$

$u = 2+2x$

$\frac{du}{2} = \cancel{2}dx$

$$\frac{\cancel{2}\pi}{\cancel{2}}\int u^{1/2}\,du$$

$$= \frac{2}{3}\pi u^{3/2} + C$$

$$= \frac{2\pi}{3}(2+2x)^{3/2}\Big]_1^7$$

$$= \frac{2\pi}{3}\left(16^{3/2} - 4^{3/2}\right)$$

$$= \frac{2\pi}{3}(64-8) = \frac{112\pi}{3}$$

OR int. wrt. y

$$\int_c^d 2\pi y\sqrt{1+(g'(y))^2}\,dy$$

$y^2 = 1+2x \rightarrow x = \frac{y^2}{2} - \frac{1}{2} = g(y)$

$g'(y) = y$ $\quad \sqrt{3} \le y \le \sqrt{15}$

$$S = \int_{\sqrt{3}}^{\sqrt{15}} 2\pi y\sqrt{1+y^2}\,dy$$

$u = 1+y^2$

$du = 2y\,dy$

$$\pi\int u^{1/2}\,du$$

$$= \frac{2\pi}{3}u^{3/2} + C$$

$$= \frac{2\pi}{3}(1+y^2)^{3/2}\Big]_{\sqrt{3}}^{\sqrt{15}}$$

$$= \frac{2\pi}{3}\left(16^{3/2} - 4^{3/2}\right)$$

$$= \frac{112\pi}{3}$$

7. Heparin, a blood-thinning drug, is an important treatment used to reduce blood clots, and its level must be very carefully monitored for safe use. We will use a differential equation model to study how the amount of heparin changes over time.

[1 mark] a) Let $H(t)$ be the amount of heparin in the body (in mg) at time t (hours). For a given patient, heparin is injected (**added**) continuously at a rate of $4mg/hr$, and the body metabolizes the drug continuously and **removes** heparin from the body at a rate proportional to H where the constant of proportionality is determined to be 0.8 with units $(1/hrs)$.

Circle the correct separable differential equation that represents the net (total) rate of change of H.

i) $\frac{dH}{dt} = 0.8H - 4$ ii) $\frac{dH}{dt} = 4 - 0.8H$ iii) $\frac{dH}{dt} = 4 + 0.8H$

[4 marks] b) Solve the separable differential equation you chose from part a) to find an explicit expression for $H(t)$, given the initial value that $H(0) = 0$ (which means that treatment is started at $t = 0$).

$\frac{dH}{dt} = 4 - 0.8H \rightarrow \int \frac{1}{4-0.8H} dH = \int (1) dt$

$-\frac{1}{0.8} \ln|4-0.8H| = t + C$

$\ln|4-0.8H| = -0.8t - 0.8C$

exponentiate $|4-0.8H| = e^{-0.8C} e^{-0.8t}$

* Can replace $e^{-0.8C}$ with a generic constant AND drop the abs. value.

$4 - 0.8H = Ae^{-0.8t}$

A could be + or − to account for $|x| = \begin{cases} +x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$

data $(H(0)=0)$ $4 - 0 = A\cancel{e^{-0.8(0)}}$

$A = 4$

$4 - 0.8H = 4e^{-0.8t}$

solve for H $H = 5 - 5e^{-0.8t}$

[2 marks] c) Find the long-term equilibrium mass of heparin in the body if this treatment were continued indefinitely. (Note: if you were unable to answer part b), this problem can be solved using only the information in part a).)

$\lim_{t\to\infty} H(t) = \lim_{t\to\infty} 5 - \underbrace{5e^{-0.8t}}_{\to 0} = 5$

OR: If the Hep. level stabalizes as $t \to \infty$, then $\frac{dH}{dt} = 0 = 4 - 0.8H$

$H = \frac{4}{0.8} = 5$

5 mg.

Over

8. Consider a curve defined by parametric equations $x = \frac{1}{\sqrt{t}}$, $y = t^2 + 1$.

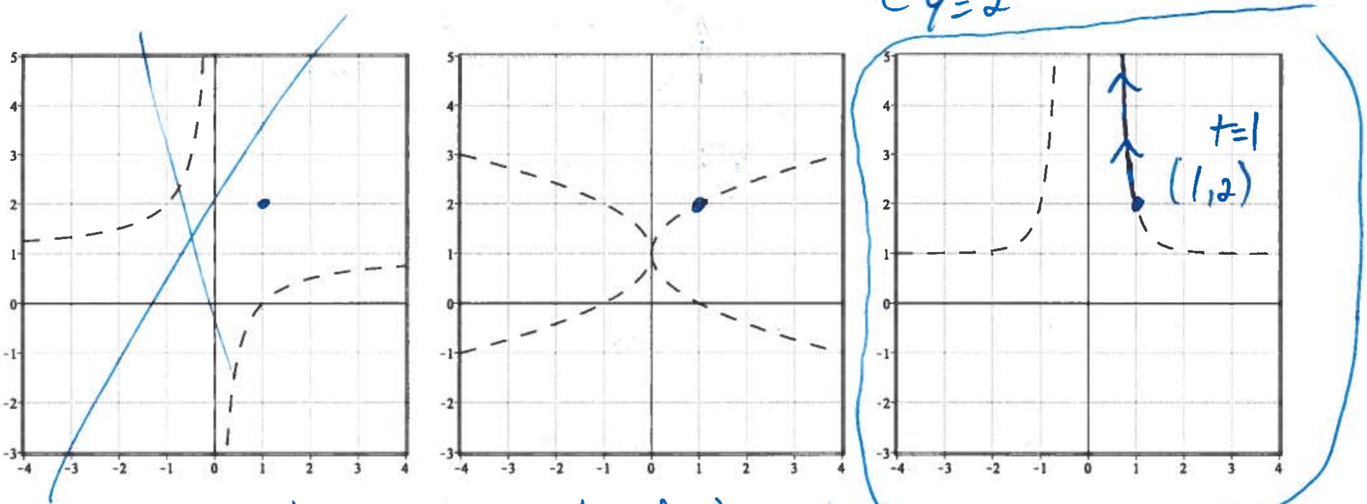
[3 marks] a) Find a Cartesian equation (in x's and y's) for the curve by eliminating the parameter t.

$x = \frac{1}{\sqrt{t}} \to x^2 = \frac{1}{t} \to t = \frac{1}{x^2}$

sub. into eq'n for y: $y = t^2 + 1 = \left(\frac{1}{x^2}\right)^2 + 1 \to$ $y = \frac{1}{x^4} + 1$

[3 marks] b) Circle the graph below which shows the shape and path of the curve, and fill in (trace with your pen/pencil) the part of the graph traversed as t increases, starting with $t = 1$. Use an arrow to indicate the direction on the graph.

$t = 1 \to \begin{cases} x = 1 \\ y = 2 \end{cases}$



as t inc., $x = \frac{1}{\sqrt{t}}$ dec's.

$y = t^2 + 1$ inc's.

[3 marks] c) Using parametric methods, find the Cartesian equation of the tangent line at the point with $t = 1$.

$x = t^{-1/2}$

$\frac{dy}{dt} = 2t$ $\qquad$ $\frac{dx}{dt} = -\frac{1}{2} t^{-3/2}$

@ $t=1$ $\frac{dy}{dt} = 2$ $\qquad$ @ $t=1$ $\frac{dx}{dt} = -\frac{1}{2}$

tan. slope $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{-1/2} = -4$

tan. line: $y = mx + b$

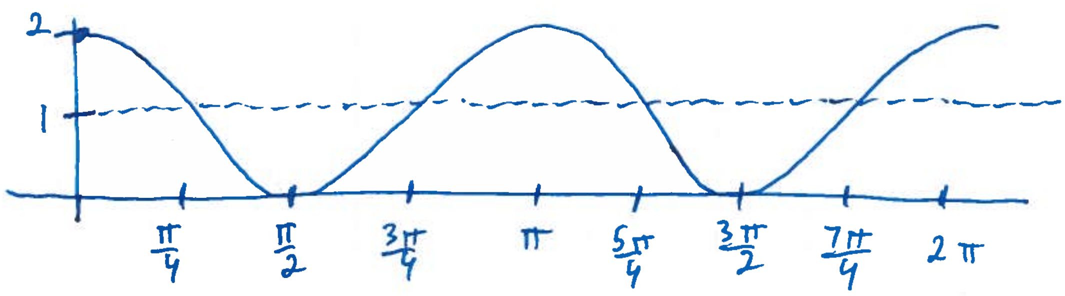
$y = -4x + b$ $\qquad$ $(1,2)$ @ $t=1$

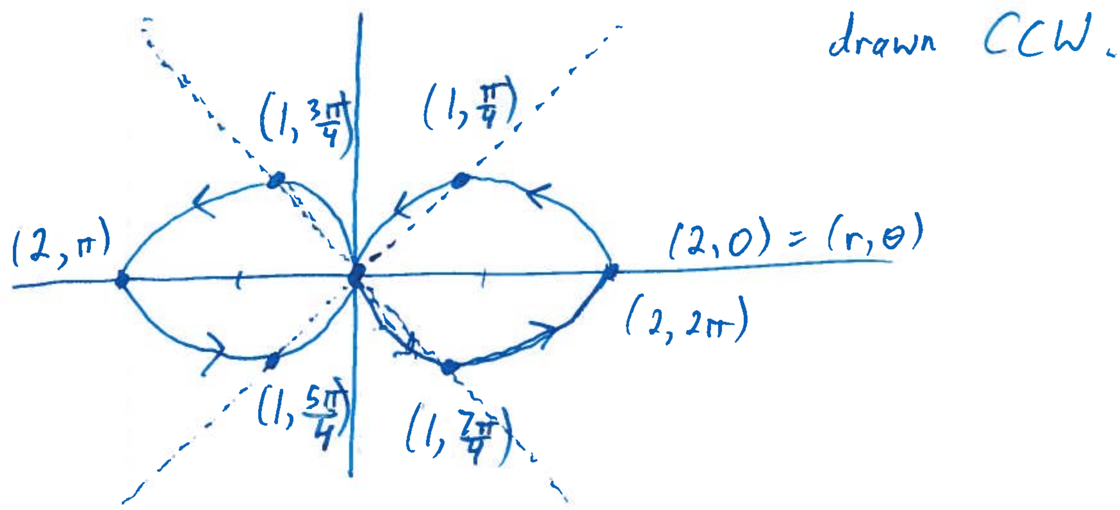
$2 = -4(1) + b \to b = 6$

$y = -4x + 6$ OR $y - 2 = -4(x-1)$

Over

[*3 marks*] **9.** a) Sketch the polar curve $r = 1 + \cos(2\theta)$.





[*2 marks*] b) Write an integral which will compute the area trapped inside the polar curve $r = 1+\cos(2\theta)$. (Hint: The curve is traced once on $0 \le \theta \le 2\pi$).

$$A = \frac{1}{2}\int_0^{2\pi} (1+\cos(2\theta))^2 \, d\theta$$

[*4 marks*] c) Find the area trapped by the polar curve $r = 1 + \cos(2\theta)$.

$$A = \frac{1}{2}\int_0^{2\pi} 1 + 2\cos(2\theta) + \cos^2(2\theta)\, d\theta$$

ID:
$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

$$= \frac{1}{2}\int_0^{2\pi} 1 + 2\cos(2\theta) + \frac{1}{2} + \frac{1}{2}\cos(4\theta)\, d\theta$$

$$= \frac{1}{2}\left(\frac{3}{2}\theta + \sin(2\theta) + \frac{\sin(4\theta)}{8}\right)\Bigg]_0^{2\pi}$$

$$= \frac{1}{2}\left(\frac{3(2\pi)}{2} + 0 + \frac{0}{8} - \left(\frac{3}{2}(0) + 0 + \frac{0}{8}\right)\right)$$

$$= \frac{3\pi}{2}$$

(Extra Space, if needed)

Trigonometric Identities

$\sin^2 x + \cos^2 x = 1$

$\sec^2 x - \tan^2 x = 1$

$\csc^2 x - \cot^2 x = 1$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

$\sin^2 x = \dfrac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \dfrac{1}{2}(1 + \cos 2x)$

$\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

$\sin(A)\cos(B) = \dfrac{1}{2}[\sin(A-B) + \sin(A+B)]$

$\sin(A)\sin(B) = \dfrac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\cos(A)\cos(B) = \dfrac{1}{2}[\cos(A-B) + \cos(A+B)]$

Derivatives of Inverse Trigonometric Functions

$\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$

$\dfrac{d}{dx}\cos^{-1} x = -\dfrac{1}{\sqrt{1-x^2}}$

$\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1+x^2}$

$\dfrac{d}{dx}\cot^{-1} x = -\dfrac{1}{1+x^2}$

$\dfrac{d}{dx}\sec^{-1} x = \dfrac{1}{x\sqrt{x^2-1}}$

$\dfrac{d}{dx}\csc^{-1} x = -\dfrac{1}{x\sqrt{x^2-1}}$

Some Integrals (not required to memorize)

$\int \ln x \, dx = x \ln x - x + C$

$\int \sec x \, dx = \ln|\sec x + \tan x| + C$

$\int \sec^3 x \, dx = \dfrac{1}{2}\sec x \tan x + \dfrac{1}{2}\ln|\sec x + \tan x| + C$