Write complete and concise proofs. For each proof, if needed, please provide clean diagrams to illustrate your proof. Marks will be deducted for messiness. The page numbers and problem numbers refer to the ones in the course pack.

Due: Sept. 19, 2018

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1. (**Page 4, #1**) Prove that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

for all $n \in \mathbb{N}$, $n > 1$.

Solution: There are many ways of proving this. One can use mathematical induction, as following.

Step 1*:* When $n = 2$, we compute the left hand side to be $1 + \frac{\sqrt{2}}{2}$ and the right hand side to be $\sqrt{2} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$. Since $1 > \frac{\sqrt{2}}{2}$, we see that left hand side is indeed greater than the right hand side for $n = 2$.

Induction hypothesis: Now assume that when $n = k > 1$ the inequality holds, i.e.

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} > \sqrt{k}$$

Induction step: We show that when $n = k+1$ the inequality holds as well, given the induction hypothesis. We compute the left hand side

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$$

where the inequality follows from the induction hypothesis. Now we compare the right hand side above, $\sqrt{k} + \frac{1}{\sqrt{k+1}}$, with the right hand side of the original inequality for $n = k+1$, i.e. $\sqrt{k+1}$. We compute their difference

$$\sqrt{k}+\frac{1}{\sqrt{k+1}}-\sqrt{k+1} = \frac{\sqrt{k(k+1)}+1-(k+1)}{\sqrt{k+1}} = \frac{\sqrt{k(k+1)}-k}{\sqrt{k+1}} = \frac{\sqrt{k}}{\sqrt{k+1}}(\sqrt{k+1}-\sqrt{k}) > 0$$

Thus we can combine the two inequalities we obtained in this step and get

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$$

Conclusion: By Mathematical Induction, we see that the inequality

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

holds for all $n \in \mathbb{N}$, $n > 1$.

Alternate solution: First we notice that for $k > 1$, we have

$$\frac{1}{\sqrt{k}} > \frac{1}{\sqrt{k}+\sqrt{k-1}} = \sqrt{k} - \sqrt{k-1}$$

Thus it follows that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > 1 + (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \ldots + (\sqrt{n} - \sqrt{n-1}) = \sqrt{n}$$

2. (**Page 4, #2**) Let the number x_n be defined as follows $x_1 := 1$, $x_2 := 2$, and $3x_{n+2} := x_{n+1} + 2x_n$ for all $n \in \mathbb{N}$. Use the Principle of Strong Induction (1.2.5) to show that $1 \leqslant x_n \leqslant 2$ for all $n \in \mathbb{N}$. (In particular, this proves that the set $\{x_1, x_2, x_3, \ldots, x_n, \ldots\}$ is bounded.)

   Solution: Let's use the Principle of Strong Induction as required.

   Step 1: It's straightforward to verify that $1 \leqslant x_1 < x_2 \leqslant 2$.

   Induction hypothesis: Assume that $1 \leqslant x_k \leqslant 2$ holds for all $1 \leqslant k < n$, for some $n > 2$.

   Induction step: We now need to prove that $1 \leqslant x_n \leqslant 2$. By definition, we have

$$x_n = \frac{1}{3}x_{n-1} + \frac{2}{3}x_{n-2} \geqslant \frac{1}{3} + \frac{2}{3} = 1$$

   and

$$x_n = \frac{1}{3}x_{n-1} + \frac{2}{3}x_{n-2} \leqslant \frac{1}{3} \times 2 + \frac{2}{3} \times 2 = 2$$

   where both inequalities uses the induction hypothesis. Thus $1 \leqslant x_n \leqslant 2$.

   Conclusion: By the Principle of Strong Induction, we see that the inequality $1 \leqslant x_n \leqslant 2$ holds for all $n \in \mathbb{N}$.

3. (**Page 4, #3**) Let $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. Prove that for all positive integers n, the following holds:

$$H_n = 1 + \frac{1}{n}\sum_{p=1}^{n-1} H_p$$

   Solution: There are at least two ways of proving this. Let's use mathematical induction.

   Step 1: When $n = 1$, both left and right hand sides are 1, which implies that the identity holds in this case.

   Induction hypothesis: Now assume that when $n = k \geqslant 1$ the identity holds, i.e.

$$H_k = 1 + \frac{1}{k}\sum_{p=1}^{k-1} H_p$$

   Induction step: We show that when $n = k + 1$ the identity holds as well, given the induction hypothesis. The left hand side now is

$$H_{k+1} = H_k + \frac{1}{k+1}$$

   The right hand side becomes

$$1 + \frac{1}{k+1}\sum_{p=1}^{k} H_p = \left(1 + \frac{1}{k}\sum_{p=1}^{k-1} H_p\right) + \frac{1}{k+1}\sum_{p=1}^{k} H_p - \frac{1}{k}\sum_{p=1}^{k-1} H_p$$

By the induction hypothesis, we only need to prove that

$$\frac{1}{k+1}\sum_{p=1}^{k} H_p - \frac{1}{k}\sum_{p=1}^{k-1} H_p = \frac{1}{k+1}$$

Clear the denominator and the above is equivalent to

$$k\sum_{p=1}^{k} H_p - (k+1)\sum_{p=1}^{k-1} H_p = k$$

We compute

$$k\sum_{p=1}^{k} H_p - (k+1)\sum_{p=1}^{k-1} H_p = kH_k - \sum_{p=1}^{k-1} H_p = k\left(H_k - \frac{1}{k}\sum_{p=1}^{k-1} H_p\right) = k$$

where the last equality used the induction hypothesis again. Thus the identity holds for $n = k+1$ as well.

Conclusion: By mathematical induction, we see that

$$H_n = 1 + \frac{1}{n}\sum_{p=1}^{n-1} H_p$$

holds for all positive integers n.

Alternate solution: The identity in the statement is equivalent to

$$H_n - \frac{1}{n}\sum_{p=1}^{n-1} H_p = 1$$

We compute

$$\begin{aligned}
& H_n - \frac{1}{n}\sum_{p=1}^{n-1} H_p \\
=& 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} \\
-& \frac{1}{n}\Big(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} \\
& + 1 + \frac{1}{2} + \ldots + \frac{1}{n-2} \\
& + \vdots \\
& + 1 + \frac{1}{2} \\
& + 1\Big) \qquad \text{sum the terms in the bracket vertically} \\
=& 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} - \frac{1}{n}\left((n-1) + \frac{n-2}{2} + \frac{n-3}{3} + \ldots + \frac{1}{n-1}\right) \\
=& 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} - \left(\frac{n-1}{n} + \frac{1}{2}\frac{n-2}{n} + \ldots + \frac{1}{n-2}\frac{2}{n} + \frac{1}{n-1}\frac{1}{n}\right) \\
=& \frac{1}{n} + \frac{1}{2}\frac{2}{n} + \frac{1}{3}\frac{3}{n} + \ldots + \frac{1}{n-1}\frac{n-1}{n} + \frac{1}{n} = n \times \frac{1}{n} = 1
\end{aligned}$$

4. (**Page 6, #1**) Solve the inequality $|x^2 - 4| \leqslant |x - 2|$.

   Solution: Factor the left hand side and we get $|x-2|(|x+2|-1) \leqslant 0$. Since $|x-2| \geqslant 0$ for all x, we see that $|x-2| = 0 \Longrightarrow x = 2$ solves the inequality as well as

   $$|x-2| \neq 0 \Longrightarrow |x+2| \leqslant 1 \Longrightarrow -1 \leqslant x+2 \leqslant 1 \Longrightarrow -3 \leqslant x \leqslant -1$$

   Thus the solution is $\{x = 2\} \cup \{-3 \leqslant x \leqslant -1\}$.

5. (**Page 6, #2**) Solve the inequality $|\sqrt{x} - 3| \leqslant \dfrac{1}{10}|x - 9|$.

   Solution: One can either work with the left hand side or factor the right hand side. Let's work with the left. Since $\sqrt{x} \geqslant 0$, $\sqrt{x} + 3 \neq 0$ for all x in the domain (which is $x \geqslant 0$). Thus the inequality is equivalent to

   $$\left|\frac{x-9}{\sqrt{x}+3}\right| \leqslant \frac{1}{10}|x-9|$$

   Since $|x-9| \geqslant 0$ for all x, we see that the above is equivalent to $x = 9$ or

   $$|\sqrt{x}+3| \geqslant 10 \iff \sqrt{x}+3 \geqslant 10 \text{ or } \sqrt{x}+3 \leqslant -10$$

   Since $\sqrt{x} \geqslant 0$, we see that $\sqrt{x} \geqslant 7 \Longrightarrow x \geqslant 49$ or $x = 9$.

6. (**Page 6, #3**) Let $\varepsilon > 0$ and $\delta > 0$, and $a \in \mathbb{R}$. Show that $V_\varepsilon(a) \cap V_\delta(a)$ and $V_\varepsilon(a) \cup V_\delta(a)$ are γ-neighbourhoods of a for appropriate values of γ.

   Solution: The key is to realize

   $$\varepsilon \leqslant \delta \iff V_\varepsilon(a) \subseteq V_\delta(a)$$

   We prove this in the following. There are two directions.
   "$\Longrightarrow$": Let $\varepsilon \leqslant \delta$, then $a-\delta \leqslant a-\varepsilon < a+\varepsilon \leqslant a+\delta$. For any $x \in V_\varepsilon(a)$ then $a-\varepsilon < x < a+\varepsilon$. It follows that $a-\delta < x < a+\delta$, which implies that $x \in V_\delta(x)$. Thus $V_\delta(a) \subseteq V_\varepsilon(a)$.
   "$\Longleftarrow$": Suppose that $V_\varepsilon(a) \subseteq V_\delta(a)$, assume that $\varepsilon > \delta$. The proof of $\Longrightarrow$ above implies that $V_\delta \subseteq V_\varepsilon$. Thus $V_\delta = V_\varepsilon$. On the other hand, since $\varepsilon > \delta$, we have $a < a+\delta < a+\varepsilon$, which implies that $a+\delta \in V_\varepsilon$ but $a+\delta \notin V_\delta$ – which contradicts $V_\delta = V_\varepsilon$. Thus assumption false and $\varepsilon \leqslant \delta$.

   Suppose that $\varepsilon \leqslant \delta$. Then $V_\varepsilon \subseteq V_\delta$ implies that

   $$V_\varepsilon(a) \cap V_\delta(a) = V_\varepsilon(a) \text{ and } V_\varepsilon(a) \cup V_\delta(a) = V_\delta(a)$$

   Thus the γ should be ε in the first case, while δ in the second case.

   In general, we have

   $$V_\varepsilon(a) \cap V_\delta(a) = V_{\min\{\varepsilon,\delta\}}(a) \text{ and } V_\varepsilon(a) \cup V_\delta(a) = V_{\max\{\varepsilon,\delta\})}(a)$$

   and it gives the appropriately chosen γ.

7. (**Page 6, #4**) Let $\varepsilon > 0$ and $a, b \in \mathbb{R}$. Suppose that $V_\varepsilon(a) \cap V_\varepsilon(b) \neq \emptyset$. Let $c \in V_\varepsilon(a) \cap V_\varepsilon(b)$, prove that there exists $\delta > 0$ such that $V_\delta(c) \subseteq V_\varepsilon(a) \cap V_\varepsilon(b)$.

   Solution: We realize that $c \in V_\varepsilon(a) \cap V_\varepsilon(b) \Longrightarrow c \in V_\varepsilon(a)$ and $c \in V_\varepsilon(b)$. Thus there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$V_{\delta_1}(c) \subseteq V_\varepsilon(a) \text{ and } V_{\delta_2}(c) \subseteq V_\varepsilon(b)$$

   Then let $\delta = \min\{\delta_1, \delta_2\}$, we see that $V_\delta(c) \subseteq V_\varepsilon(a) \cap V_\varepsilon(b)$.

   Alternate solution: $c \in V_\varepsilon(a) \cap V_\varepsilon(b)$ implies that $|c - a| < \varepsilon$ and $|b - a| < \varepsilon$. Thus we let $\delta = \min\{\varepsilon - |c - a|, \varepsilon - |b - a|\}$, then $\delta > 0$. For any $y \in V_\delta(c)$, we compute

$$|y - a| \leqslant |y - c| + |c - a| < \delta + |c - a| \leqslant \varepsilon - |c - a| + |c - a| = \varepsilon \Longrightarrow y \in V_\varepsilon(a)$$

   and

$$|y - b| \leqslant |y - c| + |c - b| < \delta + |c - b| \leqslant \varepsilon - |c - b| + |c - b| = \varepsilon \Longrightarrow y \in V_\varepsilon(b)$$

   Thus, we see that $V_\delta(c) \subseteq V_\varepsilon(a) \cap V_\varepsilon(b)$.

8. (**Page 8, #1**) Let $S \subseteq \mathbb{R}$ be a nonempty set bounded below. Prove that $\ell = \inf(S)$ iff $\ell \in L(S)$ and for any $\varepsilon > 0$, there exists $x \in S$ such that $x \in V_\varepsilon(\ell)$.

   Solution: "$\Longrightarrow$": The definition of infimum states that

   $\ell \in L(S)$ and for all $k \in L(S)$, $k \leqslant \ell$.

   Let $\varepsilon > 0$, then $\ell + \varepsilon \notin L(S)$, since $\ell + \varepsilon > \ell$. It follows that there is $x \in S$ such that $x < \ell + \varepsilon$. Since $\ell \in L(S)$, we see that $x \geqslant \ell$. Thus $\ell - \varepsilon < \ell \leqslant x < \ell + \varepsilon$, i.e. $x \in V_\varepsilon(\ell)$.

   "$\Longleftarrow$": We verify the definition. $\ell \in L(S)$ holds, i.e. $x \geqslant \ell$ for all $x \in S$. Let $k \in L(S)$ and assume that $k > \ell$. Let $\varepsilon = k - \ell > 0$, then there is $x \in S$ such that $x \in V_\varepsilon(\ell)$, i.e. $\ell - \varepsilon < x < \ell + \varepsilon = k$. This contradicts $k \in L(S)$. Thus we must have $k \leqslant \ell$.

9. (**Page 8, #3**) Suppose that $S \subseteq \mathbb{R}$ is a nonempty set bounded above. Let $-S = \{x \in \mathbb{R} : -x \in S\}$. Prove that $-\sup(S) = \inf(-S)$.

   Solution: Let $u = \sup(S)$, i.e. $u \geqslant x$ for all $x \in S$ and for any $\varepsilon > 0$, there exists $y \in S$ and $y \in V_\varepsilon(u)$. Then for any $x' \in -S$, we have $-x' \in S$. Thus $u \geqslant -x'$, i.e. $-u \leqslant x'$. It follows that $-u$ is a lower bound of $-S$, in particular, $-S$ is bounded below.

   For any $\varepsilon > 0$, since $u = \sup(S)$, there is $y \in S$ and $y \in V_\varepsilon(u)$, i.e. $|y - u| < \varepsilon$. Then $-y \in -S$ and we have $|(-y) - (-u)| = |y - u| < \varepsilon$. It follows that $-u = \inf(-S)$, i.e. $-\sup(S) = \inf(-S)$.

   Alternate proof: Let $u = \sup(S)$. For any $x \in -S$, we see that $-x \in S$, which implies that $-x \leqslant u$ and $x \geqslant -u$. Thus $\inf(-S) \geqslant -u = -\sup(S)$.

   Let $v = \inf(-S)$. For any $y \in S$, we see that $-y \in -S$, which implies that $-y \geqslant v$ and $y \leqslant -v$. Thus $\sup(S) \leqslant -v = -\inf(-S)$, which gives $\inf(-S) \leqslant -\sup(S)$.

   The two inequalities combine give $-\sup(S) = \inf(-S)$.

10. (**Page 8, #4**) Prove that the following two statements about the completeness of $\mathbb{R}$ is equivalent.

(a) Any nonempty subset of $\mathbb{R}$ that is bounded above has a supremum.

(b) Any nonempty subset of $\mathbb{R}$ that is bounded below has an infimum.

Solution: $(a) \Longrightarrow (b)$: Let $S \subseteq \mathbb{R}$ be bounded below and let $L \in \mathbb{R}$ be one of the low bounds of S, i.e.

$$\forall x \in S, x \geqslant L$$

Set $T = -S = \{y \in \mathbb{R} : -y \in S\}$, then for any $y \in T$, we have

$$y = -x \text{ for some } x \in S \Longrightarrow x \geqslant L \Longrightarrow y = -x \leqslant -L$$

Thus $-L$ is a upper bound of T. By (a), we see that T has a supremum, say $u = \sup(T)$. Then

$$\forall x \in S \Longrightarrow -x \in T \Longrightarrow -x \leqslant u \Longrightarrow x \geqslant -u$$

thus $-u$ is a lower bound of S. We prove that $-u = \inf(S)$ and thus prove (b). Since $u = \sup(T)$, we see that

$$\forall \varepsilon > 0, \exists y \in T, \text{ such that } y \in V_\varepsilon(u)$$

Take this y, then we have $y = -x$ for some $x \in S$ and we see that

$$-x \in V_\varepsilon(u) \iff u - \varepsilon < -x < u + \varepsilon \iff -u - \varepsilon < x < -u + \varepsilon \iff x \in V_\varepsilon(-u)$$

Thus $-u = \inf(S)$.

$(b) \Longrightarrow (a)$: Essentially similar to the above, and make sure that you can write it up correctly.

11. (**Page 9, #5**) Let S be a (nonempty) bounded set in $\mathbb{R}$ and let S_0 be a nonempty subset of S. Show that

$$\inf(S) \leqslant \inf(S_0) \leqslant \sup(S_0) \leqslant \sup(S).$$

Solution: Let $\ell = \inf(S)$, then for any $x \in S_0 \subseteq S$, we have $\ell \leqslant x$, i.e. ℓ is a lower bound of S_0. By definition of infimum, we see that $\ell \leqslant \inf(S_0)$.

Let $u = \sup(S)$, then for any $x \in S_0 \subseteq S$, we have $x \leqslant u$, i.e. u is an upper bound of S_0. By definition of supremum, we see that $\sup(S_0) \leqslant u$.

By definition of inf and sup, we see that for $x \in S_0 \neq \emptyset$, we get $\inf(S_0) \leqslant x \leqslant \sup(S_0)$. Thus

$$\inf(S) \leqslant \inf(S_0) \leqslant \sup(S_0) \leqslant \sup(S).$$

12. (**Page 9, #6**) Let $S \subseteq \mathbb{R}$ and suppose that $s^* := \sup(S)$ belongs to S. If $u \notin S$, show that $\sup(S \cup \{u\}) = \sup\{s^*, u\}$.

Solution: Since $\{s^*, u\} \subseteq S \cup \{u\}$, the previous problem implies that

$$\sup(S \cup \{u\}) \geqslant \sup\{s^*, u\}$$

There are many ways to deal with the rest of the proof, one of which is the following.

Since $s^* \in S$ but $u \notin S$, we have $u \neq s^*$. Suppose that $u > s^*$, then u is an upper bound of S and $u \geqslant u$. Thus u is an upper bound of $S \cup \{u\}$. It follows that

$$\sup(S \cup \{u\}) \leqslant u \leqslant \sup\{s^*, u\} \Longrightarrow \sup(S \cup \{u\}) = \sup\{s^*, u\}$$

On the other hand, suppose that $u < s^*$, then s^* is an upper bound of $S \cup \{u\}$. Thus

$$\sup(S \cup \{u\}) \leqslant s^* \leqslant \sup\{s^*, u\} \Longrightarrow \sup(S \cup \{u\}) = \sup\{s^*, u\}$$

(In fact, we could say that $\max\{s^*, u\}$ is an upper bound of $S \cup \{u\}$ and go with that).

13. (**Page 9, #7**) Let S be a nonempty bounded set in $\mathbb{R}$. Let $L(S)$ denote the set of lower bounds of S. Prove that $\inf(S) = \sup(L(S))$.

    Solution: First, since any $x \in S$ and $y \in L(S)$, we have $x \geqslant y$, we see that $\inf(S) \geqslant \sup(L(S))$. We only have to prove now $\inf(S) \leqslant \sup(L(S))$. Since $\inf(S) \in L(S)$, we see that $\inf(S) \leqslant \sup(L(S))$ by definition of supremum. Thus $\inf(S) = \sup(L(S))$.

14. (**Page 11, #4**) If $u > 0$ is any real number and $x < y$, show that there exists a rational number r such that $x < ru < y$.

    Solution: Consider $x' = \dfrac{x}{u}$ and $y' = \dfrac{y}{u}$, since $u > 0$ we see that $x' < y'$. By the density of rational numbers, there thus exists a rational number r such that $x' < r < y'$. It follows that $x < ru < y$.
