Write complete and concise proofs. For each proof, if needed, please provide clean diagrams to illustrate your proof. Marks will be deducted for messiness. The page numbers and problem numbers refer to the ones in the course pack.

Due: Oct. 17, 2018

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1. (**Page 11, #1**) Do not use the Archimedean property, prove directly from the Completeness of $\mathbb{R}$, that the set $\{\sqrt{n} : n \in \mathbb{N}\}$ is unbounded.

   Solution: Proof by contradiction. Assume that $\{\sqrt{n} : n \in \mathbb{N}\}$ is bounded, i.e. it has an upper bound $M \in \mathbb{R}$. By the Completeness of $\mathbb{R}$, we see that the set $\{\sqrt{n} : n \in \mathbb{N}\}$ has a supremum, say it is u. Then we have

   (a) $\sqrt{n} \leqslant u$ for all $n \in \mathbb{N} \Longrightarrow n \leqslant u^2$, for all $n \in \mathbb{N}$, and

   (b) in particular, $u^2 \geqslant 2 > 1$. Thus for $v = \sqrt{u^2 - 1} < u$, there must be $m \in \mathbb{N}$ such that $\sqrt{m} > v = \sqrt{u^2 - 1}$.

   For this particular m, we have $m > u^2 - 1 \Longrightarrow u^2 < m + 1 \in \mathbb{N}$. This is contradiction to (a) above. Thus the assumption was wrong and the set $\{\sqrt{n} : n \in \mathbb{N}\}$ is unbounded.

2. (**Page 11, #2**) If $y > 0$, show that there exists $n \in \mathbb{N}$ such that $\dfrac{1}{n!} < y$.

   Solution: Prove by contradiction. Assume that for all $n \in \mathbb{N}$, $\dfrac{1}{n!} \geqslant y$, it then implies that $n! \leqslant \dfrac{1}{y}$ for all $n \in \mathbb{N}$, in particular, $\{n! : n \in \mathbb{N}\}$ is bounded above by $\dfrac{1}{y}$.

   Let $u = \sup\{n! : n \in \mathbb{N}\}$, then $u \geqslant n!$ for all $n \in \mathbb{N}$, in particular $u > 0$. Since $u > \dfrac{u}{2}$, there exists $k \in \mathbb{N}$ such that $k! > \dfrac{u}{2}$. Thus $(k+2)! = (k+2)(k+1)k! > u$ with $k + 2 \in \mathbb{N}$, which is contradiction.

   Thus the assumption is false and there exists $n \in \mathbb{N}$ such that $\dfrac{1}{n!} < y$.

3. (**Page 11, #3**) Let $S = \left\{2 - \dfrac{1}{n} : n \in \mathbb{N}\right\}$. Find $\sup(S)$.

   Solution: Since $n \in \mathbb{N} \Longrightarrow n > 0$, we see that

$$x \in S \Longrightarrow x = 2 - \frac{1}{n} \text{ for some } n \in \mathbb{N} \Longrightarrow x < 2$$

   Thus 2 is an upper bound of S and by the Completeness of $\mathbb{R}$, $\sup(S)$ exists.

   For any $\varepsilon > 0$, by the Archimedean property, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \varepsilon \Longrightarrow x = 2 - \frac{1}{n} \in S \text{ satisfies } 2 - \varepsilon < x \Longrightarrow x \in V_\varepsilon(2)$$

   Thus $\sup(S) = 2$.

4. (**Page 11, #5**) Prove that for any x in a complete ordered field K, there is $n \in \mathbb{Z}$ such that $n-1 \leqslant x \leqslant n$. (This is Corollary 3.30 in the notes.)

   Solution: By the Archimedean property, we see that there is $m \in \mathbb{Z}$ such that $x \leqslant m$. Similarly, there is $l \in \mathbb{Z}$ such that $-x \leqslant -l \Longrightarrow l \leqslant x$. Thus, let

   $$L_x := \{l \in \mathbb{Z} : l < x\} \text{ and } U_x := \{m \in \mathbb{Z} : x \leqslant m\}$$

   then L_x is bounded above and U_x be bounded below, and $\forall m \in U_x$ and $\forall l \in L_x$, we have $l < m$. Choose $l_0 \in L_x$, then

   $$U_x - l_0 := \{m - l_0 : m \in U_x\} \subseteq \mathbb{N}$$

   which has a first element — the least element — because $\mathbb{N}$ is well-ordered. Let d be the least element of $U_x - l_0$. Then $n = d + l_0$ is the least element of U_x.

   Since integers are either small than x — in L_x or not smaller than x — in U_x, we see that $n-1$ must be an integer in L_x. It follows that

   $$n-1 < x \leqslant n \Longrightarrow n-1 \leqslant x \leqslant n$$

5. (**Page 13, #1**) Let $a \in \mathbb{R}$ and $S \subseteq \mathbb{R}$ be a non-empty subset, which is bounded above. Prove that $\sup(a+S) = a + \sup(S)$.

   Solution: Let $u = \sup(A)$ and $v = \sup(B)$, then for all $a \in A$ and $b \in B$ we have $a \leqslant u$ and $b \leqslant v$. It follows that for any $s \in A+B$, there are $a \in A$ and $b \in B$, such that $s = a+b \leqslant u+v$. Thus $A+B$ is bounded above and

   $$\sup(A+B) \leqslant u+v = \sup(A) + \sup(B)$$

   For any $\varepsilon > 0$ consider $K = u+v-\varepsilon$. We would like to show that K is not an upper bound of $A+B$. In fact, by definition of supremum, $u - \frac{\varepsilon}{2}$ is no an upper bound of A, i.e. there is $a \in A$ such that $a > u - \frac{\varepsilon}{2}$. Similarly, there is $b \in B$ such that $b > v - \frac{\varepsilon}{2}$. It follows that for these a, b, we have

   $$a+b \in A+B \text{ and } a+b > u+v-\varepsilon = K$$

   i.e. K is not an upper bound of $A+B$.

   This implies that $u+v$ is the supremum of $A+B$, i.e. $\sup(A+B) = \sup(A) + \sup(B)$.

   The part on inf is similar, and you should make sure that you can write it out, without using the part on sup.

   Alternate solution: Let $u = \sup(A)$ and $v = \sup(B)$, then for all $a \in A$ and $b \in B$ we have $a \leqslant u$ and $b \leqslant v$. It follows that for any $s \in A+B$, there are $a \in A$ and $b \in B$, such that $s = a+b \leqslant u+v$. Thus $A+B$ is bounded above and

   $$\sup(A+B) \leqslant u+v = \sup(A) + \sup(B)$$

   Now we compute. Let $w = \sup(A+B)$, then

   $$w \geqslant a+b \text{ for all } a \in A, b \in B \Longrightarrow w-a \geqslant b \text{ for all } b \in B \text{ and any fixed } a \in A$$

$$\Longrightarrow w - a \geqslant \sup(B) = v \text{ for any fixed } a \in A \Longrightarrow w - v \geqslant a \text{ for all } a \in A$$
$$\Longrightarrow w - v \geqslant \sup(A) = u \Longrightarrow \sup(A + B) \geqslant \sup(A) + \sup(B)$$

The two inequalities combine give $\sup(A + B) = \sup(A) + \sup(B)$.

The part on inf is similar, and you should make sure that you can write it out, without using the part on sup.

6. (**Page 13, #3**) Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that $\sup(A+B) = \sup(A)+\sup(B)$ and $\inf(A+B) = \inf(A)+\inf(B)$.

Solution: Let $u = \sup(A)$ and $v = \sup(B)$, then for all $a \in A$ and $b \in B$ we have $a \leqslant u$ and $b \leqslant v$. It follows that for any $s \in A+B$, there are $a \in A$ and $b \in B$, such that $s = a+b \leqslant u+v$. Thus $A + B$ is bounded above and

$$\sup(A + B) \leqslant u + v = \sup(A) + \sup(B)$$

For any $\varepsilon > 0$ consider $K = u + v - \varepsilon$. We would like to show that K is not an upper bound of $A + B$. In fact, by definition of supremum, $u - \dfrac{\varepsilon}{2}$ is no an upper bound of A, i.e. there is $a \in A$ such that $a > u - \dfrac{\varepsilon}{2}$. Similarly, there is $b \in B$ such that $b > v - \dfrac{\varepsilon}{2}$. It follows that for these a, b, we have

$$a + b \in A + B \text{ and } a + b > u + v - \varepsilon = K$$

i.e. K is not an upper bound of $A + B$.

This implies that $u + v$ is the supremum of $A + B$, i.e. $\sup(A + B) = \sup(A) + \sup(B)$.

The part on inf is similar, and you should make sure that you can write it out, without using the part on sup.

Alternate solution: Let $u = \sup(A)$ and $v = \sup(B)$, then for all $a \in A$ and $b \in B$ we have $a \leqslant u$ and $b \leqslant v$. It follows that for any $s \in A + B$, there are $a \in A$ and $b \in B$, such that $s = a + b \leqslant u + v$. Thus $A + B$ is bounded above and

$$\sup(A + B) \leqslant u + v = \sup(A) + \sup(B)$$

Now we compute. Let $w = \sup(A + B)$, then

$$w \geqslant a + b \text{ for all } a \in A, b \in B \Longrightarrow w - a \geqslant b \text{ for all } b \in B \text{ and any fixed } a \in A$$

$$\Longrightarrow w - a \geqslant \sup(B) = v \text{ for any fixed } a \in A \Longrightarrow w - v \geqslant a \text{ for all } a \in A$$
$$\Longrightarrow w - v \geqslant \sup(A) = u \Longrightarrow \sup(A + B) \geqslant \sup(A) + \sup(B)$$

The two inequalities combine give $\sup(A + B) = \sup(A) + \sup(B)$.

The part on inf is similar, and you should make sure that you can write it out, without using the part on sup.

7. (**Page 13, #4**) Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $AB := \{ab : a \in A, b \in B\}$. Suppose that $\sup(A) \leqslant 0$ and $\sup(B) \leqslant 0$. Prove that $\inf(AB) = \sup(A)\sup(B)$.

   Solution: Let $u = \sup(A)$ and $v = \sup(B)$, then $\forall a \in A$ and $\forall b \in B$, we have $a \leqslant u \leqslant 0$ and $b \leqslant v \leqslant 0$. It follows that $ab \geqslant uv \geqslant 0$ for all $a \in A$ and $b \in B$. Thus $\inf(AB) \geqslant uv = \sup(A)\sup(B)$ by definition.

   For any $\varepsilon > 0$, consider $K_\varepsilon = uv + \varepsilon$, then we would like to show that K_ε is not a lower bound of AB. By definition of supremum, we see that for any $\delta > 0$

   $$\exists a \in A \text{ such that } a > u - \delta \text{ and } \exists b \in B \text{ such that } b > v - \delta$$

   For such a, b, since $a, b \leqslant 0$, we have

   $$ab < (u-\delta)(v-\delta) = uv - (u+v)\delta + \delta^2$$

   If $\delta > 0$ can be chosen for any given $\varepsilon > 0$, such that $uv - (u+v)\delta + \delta^2 < uv + \varepsilon$, or equivalently

   $$-(u+v)\delta + \delta^2 < \varepsilon$$

   then we are done. The inequality is equivalently to $\delta^2 - (u+v)\delta - \varepsilon < 0$. We compute the discriminant and solve the inequality

   $$\Delta = (u+v)^2 + 4\varepsilon > |u+v| \geqslant 0 \Longrightarrow -\sqrt{\Delta} + (u+v) < \delta < \sqrt{\Delta} + (u+v)$$

   Since $\sqrt{\Delta} + (u+v) > 0$ by construction, let $\delta_0 = \dfrac{1}{2}(\sqrt{\Delta} + (u+v)) > 0$, and choose $a \in A$ and $b \in B$ such that $a > u - \delta_0$ and $b > v - \delta_0$, we have $ab > uv + \varepsilon = K_\varepsilon$, i.e. K_ε is not a lower bound of AB. By definition, we see that $\inf(AB) = uv = \sup(A)\sup(B)$.

8. (**Page 13, #5**) Let X be a nonempty set, and let f and g be defined on X and have bounded ranges in $\mathbb{R}$. Show that

   $$\sup\{f(x) + g(x) : x \in X\} \leqslant \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

   and that

   $$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leqslant \inf\{f(x) + g(x) : x \in X\}$$

   Give examples to show that each of these inequalities can be either equalities or strict inequalities.

   Solution: Write $A = \{f(x) : x \in X\}$ and $B = \{g(x) : x \in X\}$, then $\{f(x) + g(x) : x \in X\} \subseteq A + B$. It follows that

   $$\sup\{f(x)+g(x) : x \in X\} \leqslant \sup(A+B) = \sup(A)+\sup(B) = \sup\{f(x) : x \in X\}+\sup\{g(x) : x \in X\}$$

   The inf part is similar and you should make sure that you can write it out.

   For examples for the sup inequality, we consider the following.

   - "=": $X = [1, 2]$, $f(x) = x$ and $g(x) = 2x$, then $f(x) + g(x) = 3x$.

     $$\sup\{f(x) + g(x) : x \in X\} = 6, \sup\{f(x) : x \in X\} = 2 \text{ and } \sup\{g(x) : x \in X\} = 4$$

- "<": $X = [1, 2]$, $f(x) = -x$ and $g(x) = 2x$, then $f(x) + g(x) = x$.

$$\sup\{f(x) + g(x) : x \in X\} = 2, \sup\{f(x) : x \in X\} = -1 \text{ and } \sup\{g(x) : x \in X\} = 4$$

The inf examples is similar and you should make sure that you can write it out.

9. (**Page 16, #1**) Let I be an interval and suppose that $x \in I$ while $y \notin I$. Suppose further that $x > y$. Prove that y is a lower bound of I.

Solution: Follow the proof of Lemma 3.44 in the course pack.

Assume that $x > y$ and y is not a lower bound of I, then there is $z \in I$ and $z < y$. By definition of intervals, we see that $x > y > z$ with $x, z \in I$ implies that $y \in I$. Contradiction. Thus y is a lower bound of I.

10. (**Page 16, #2**) Carry out the **Alternate proof** of "$\Longrightarrow$" in Lemma 3.46 in the course pack for the following case:

> Prove that if an interval I is open and bounded then $\forall x \in I$, $\exists \varepsilon > 0$ such that $V_\varepsilon(x) \subseteq I$.

The proof also provides the justification of bounded open interval must be of the form (a, b) with $a, b \in \mathbb{R}$.

Solution: Follow the proof given in the course pack for the Alternate proof.

Since I is bounded, let $u = \sup(I)$ and $\ell = \inf(I)$. Then $\ell, u \notin I$. It follows that $\forall x \in I$, $\ell < x < u$. Let $\delta = \min(x - \ell, u - x) > 0$, then by the properties of supremum and infimum, there exists $a, b \in I$ such that

$$\ell < a < \ell + \delta \leqslant x \text{ and } x \leqslant u - \delta < b < u$$

Since I is an interval, we see that $\forall y$ with $a \leqslant y \leqslant b$ we get $y \in I$. Let $\varepsilon = \min(b - x, x - a)$, then

$$z \in V_\varepsilon(x) \Longrightarrow |z - x| < \varepsilon \Longrightarrow a < y < b \Longrightarrow y \in I$$

Thus $V_\varepsilon(x) \subseteq I$.

11. (**Page 16, #4**) Let I and I' be closed intervals in $\mathbb{R}$. Prove that $I \subseteq I'$ if and only if $\inf(I') \leqslant \inf(I)$ and $\sup(I) \leqslant \sup(I')$. (**Note:** Use the definition of closed interval in the course pack.)

Solution: "$\Longrightarrow$": Since $I \subseteq I'$, we see that $\inf(I) \in I \Longrightarrow \inf(I) \in I'$ and $\inf(I') \leqslant \inf(I)$. Similarly, $\sup(I) \in I \Longrightarrow \sup(I) \in I'$ and $\sup(I') \geqslant \sup(I)$.

"$\Longleftarrow$": Since $\inf(I') \leqslant \inf(I) \leqslant \sup(I) \leqslant \sup(I')$ and $\inf(I), \sup(I) \in I$, $\inf(I'), \sup(I') \in I'$; because I' is an interval, we see that $\inf(I) \in I'$ and $\sup(I) \in I'$. For any $x \in I$, we have $\inf(I) \leqslant x \leqslant \sup(I)$. Again, since I' is an interval, we see that $x \in I'$. Thus $I \subseteq I'$.

12. (**Page 18, #1**) Let $I_n = \left[0, \dfrac{1}{n^2}\right]$ for $n \in \mathbb{N}$. Prove that $\bigcap\limits_{n=1}^{\infty} I_n = \{0\}$.

Solution: Archimedean property implies that $\inf\left\{\dfrac{1}{n^2} - 0\right\} = 0$. Since $0 < \dfrac{1}{(n+1)^2} < \dfrac{1}{n^2}$ for all $n \in \mathbb{N}$, $\{I_n\}$ form nested intervals. By the Nested Interval Property, we see that

$\bigcap_{n=1}^{\infty} I_n = \{\xi\}$ for some unique $\xi \in \mathbb{R}$. We only need to now notice that for all n, $0 \in I_n$, which implies that $\xi = 0$ by uniqueness.

13. (**Page 18, #2**) Let $a_1, a_2, a_3, \ldots$ be a sequence of integers, all between 1 and 8, and let q_n denote the rational number $0.a_1a_2\ldots a_n$. Prove that there exists a unique real number r such that $|r - q_n| < 10^{-n}$ for all $n = 1, 2, \ldots$.

    Solution: Let $r_0 = 0$ and $u_n = 0.a_1a_2\ldots a_{n-1}9$, then by definition,

    $$r_{n-1} < r_n < u_n$$

    Thus we have $r_n \in [r_{n-1}, u_n]$. Let $I_n = [r_{n-1}, u_n]$, then they form a sequence of nested intervals, since for all $n = 1, 2, \ldots$:

    $$0 < a_n < 9 \Longrightarrow u_{n+1} = 0.a_1a_2\ldots a_{n-1}a_n9 < 0.a_1a_2\ldots a_{n-1}9 = u_n$$

    which implies that $I_{n+1} \subset I_n$. The length of the interval I_n is given by

    $$|I_n| = u_n - r_{n-1} = 9 \times 10^{-n}$$

    Let $\ell = \inf\{|I_n| : n \in \mathbb{N}\}$, we should show that $\ell = 0$. Since $|I_n| \geqslant 0$, we see that $\ell \geqslant 0$. On the other hand, assume that $\ell > 0$, then the Archimedean property implies that there is an positive integer N such that $\frac{1}{N} < \ell$. Let k be the number of digits in N, then $10^{k+1} > 10N$, which implies that

    $$\ell > \frac{1}{N} > \frac{9}{10N} > 9 \times 10^{-k-1} = |I_{k+1}|$$

    which contradicts $\ell = \inf\{|I_n| : n \in \mathbb{N}\}$. Thus $\ell = 0$.

    By the Nested Interval Property, we see that there is a unique real number r contained in all the intervals I_n. For each $n = 1, 2, \ldots$, we have $r \in I_{n+1} = [r_n, u_{n+1}]$, which implies that

    $$|r - r_n| \leqslant |I_{n+1}| = 9 \times 10^{-(n+1)} < 10^{-n}$$

14. (**Page 19, #1**) Let (x_n) be a bounded sequence. Prove that $\exists M > 0$ such that $|x_n| < M$ for all $n \in \mathbb{N}$.

    Solution: (x_n) is bounded implies that there are real numbers a and b such that $a \leqslant x_n \leqslant b$ for all $n \in \mathbb{N}$. Let $M = \max\{|a| + 1, |b| + 1\}$, then $M > 0$, $-M < -|a| \leqslant a$ and $M > |b| \geqslant b$. It follows that $-M < x_n < M$, i.e. $|x_n| < M$ for all $n \in \mathbb{N}$.

15. (**Page 25, #4**) Prove that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$. Give an example to show that the convergence of $(|x_n|)$ need not imply the convergence of (x_n).

    Solution: "$\Longrightarrow$:" $\lim(x_n) = 0$ iff

    $$\forall \varepsilon > 0,\ \exists K \in \mathbb{N}, \text{ such that } \forall n > K,\ |x_n - 0| = |x_n| < \varepsilon.$$

    For the same K, we have $||x_n| - 0| = |x_n| < \varepsilon$ for all $n > K$. By definition, this implies that $\lim(|x_n|) = 0$.

    "$\Longleftarrow$:" $\lim(|x_n|) = 0$ iff

$$\forall \varepsilon > 0, \exists K \in \mathbb{N}, \text{ such that } \forall n > K, \ ||x_n| - 0| = |x_n| < \varepsilon.$$

For the same K, we have $|x_n - 0| = |x_n| < \varepsilon$ for all $n > K$. By definition, this implies that $\lim(x_n) = 0$.

The example for non-convergence of (x_n) with $(|x_n|)$ converging is $(x_n = (-1)^n)$. The proof that convergence fails is given in

16. (**Page 25, #1**) Use the definition of the limit of a sequence to establish the following limits.

$$\lim\left(\frac{n^2-1}{3n^2+2}\right) = \frac{1}{3}$$

Solution: We need to do the following

$$\forall \varepsilon > 0, \textit{ find } K \in \mathbb{N}, \text{ such that } \forall n > K, \left|\frac{n^2-1}{3n^2+2} - \frac{1}{3}\right| < \varepsilon$$

Start from $\left|\frac{n^2-1}{3n^2+2} - \frac{1}{3}\right| < \varepsilon$, we compute:

$$\left|\frac{n^2-1}{3n^2+2} - \frac{1}{3}\right| < \varepsilon \iff \left|\frac{3(n^2-1)-(3n^2+2)}{3(3n^2+2)}\right| = \left|\frac{5}{3(3n^2+2)}\right| < \varepsilon$$

$$\Longleftarrow \frac{5}{n^2} < \varepsilon \Longleftarrow n > \sqrt{\frac{5}{\varepsilon}}$$

Thus, let $K = \left\lfloor \sqrt{\frac{5}{\varepsilon}} \right\rfloor + 1$, then for all $n > K$, we get

$$n > \sqrt{\frac{5}{\varepsilon}} \Longrightarrow \left|\frac{n^2-1}{3n^2+2} - \frac{1}{3}\right| < \varepsilon$$

which by definition, implies that $\lim\left(\frac{n^2-1}{3n^2+2}\right) = \frac{1}{2}$.

17. (**Page 25, #2**) Use the definition to prove that

$$\lim\left(\frac{\sqrt{n}+1}{n+1}\right) = 0$$

Solution: We need to do the following

$$\forall \varepsilon > 0, \textit{ find } K \in \mathbb{N} \text{ such that } \forall n > K, \left|\frac{\sqrt{n}+1}{n+1} - 0\right| < \varepsilon$$

Start from $\left|\frac{\sqrt{n}+1}{n+1} - 0\right| < \varepsilon$, we compute

$$\left|\frac{\sqrt{n}+1}{n+1} - 0\right| < \varepsilon \iff \frac{\sqrt{n}+1}{n+1} < \varepsilon \Longleftarrow \frac{2\sqrt{n}}{n} = \frac{2}{\sqrt{n}} < \varepsilon \Longleftarrow n > \frac{4}{\varepsilon^2}$$

Thus, let $K = \left\lfloor \frac{4}{\varepsilon^2} \right\rfloor + 1$, then for all $n > K$, we get

$$n > \frac{4}{\varepsilon^2} \Longrightarrow \left| \frac{\sqrt{n}+1}{n+1} - 0 \right| < \varepsilon$$

which by definition, implies that $\lim \left(\frac{\sqrt{n}+1}{n+1} \right) = 0$.

18. (**Page 25, #3**) Use the definition to prove that $\lim(\sqrt{n^2+2} - n) = 0$.

Solution: We need to do the following

$$\forall \varepsilon > 0, \textit{ find } K \in \mathbb{N} \text{ such that } \forall n > K, \left| \sqrt{n^2+2} - n - 0 \right| < \varepsilon$$

Start from $\left| \sqrt{n^2+2} - n - 0 \right| < \varepsilon$, we compute

$$\left| \sqrt{n^2+2} - n - 0 \right| < \varepsilon \iff \sqrt{n^2+2} - n < \varepsilon \iff \frac{2}{\sqrt{n^2+2}+n} < \varepsilon$$

$$\Longleftarrow \frac{2}{\sqrt{n^2}+n} = \frac{2}{2n} < \varepsilon \Longleftarrow n > \frac{1}{\varepsilon}$$

Thus, let $K = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1$, then for all $n > K$, we get

$$n > \frac{1}{\varepsilon} \Longrightarrow \left| \sqrt{n^2+2} - n - 0 \right| < \varepsilon$$

which by definition, implies that $\lim \left(\sqrt{n^2+2} - n \right) = 0$.

19. (**Page 25, #5**) Let (x_n) be a sequence. Prove or disprove the following:

(a) If $\lim(x_n) = 0$, then $\lim(|x_n| + x_n) = 0$.

(b) If $\lim(|x_n| + x_n) = 0$, then $\lim(x_n) = 0$.

Solution:

(a) This is true. By definition, $\lim(x_n) = 0$ implies that

$$\forall \varepsilon > 0, \exists K \in \mathbb{N}, \text{ such that } \forall n > K, |x_n| < \frac{\varepsilon}{2}$$

For the same K, we have for all $n > k$,

$$||x_n| + x_n| \leqslant 2|x_n| < \varepsilon$$

By definition, we see that $\lim(|x_n| + x_n) = 0$.

(b) This is false. Let $x_n = -n$, then $|x_n| = n$ and $|x_n| + x_n = 0$. Thus $\lim(|x_n| + x_n) = 0$. By Archimedean property, $(x_n = -n)$ is not bounded, thus cannot have limit.

20. (**Page 25, #6**) If $\lim(x_n) = x < 0$. Show that there exists a natural number K such that if $n \geqslant K$, then $\frac{1}{2}x > x_n > 2x$.

Solution: $\lim(x_n) = x < 0$ implies that

$$\forall \varepsilon > 0, \exists M \in \mathbb{N}, \text{ such that } \forall n > M, |x_n - x| < \varepsilon, \text{ i.e. } x - \varepsilon < x_n < x + \varepsilon.$$

Since $x < 0$, we take $\varepsilon = -x$ in the above and obtain $M_1 \in \mathbb{N}$ such that

$$\forall n > M_1, 2x \leqslant x + x < x_n < x - x, \text{ in particular } n > M_1 \Longrightarrow x_n > 2x.$$

Take $\varepsilon = -\frac{x}{2}$ in the definition, we obtain $M_2 \in \mathbb{N}$ such that

$$\forall n > M_2, x + \frac{x}{2} < x_n < x - \frac{x}{2}, \text{ in particular } n > M_2 \Longrightarrow \frac{x}{2} > x_n$$

Take now $K = \max\{M_1, M_2\} + 1$, then for all $n \geqslant K$, we see $n > M_1$ and $n > M_2$, which gives precisely

$$\frac{1}{2}x > x_n > 2x$$

21. (**Page 28, #1**) Let X and Y be two sequences. Determine if the following statements are true or false. Fully justify your answer.

(a) If $X + Y$ is convergent, then X and Y are convergent.

(b) If XY is convergent and X is convergent, then Y is convergent.

Solution: Both are false. Counterexamples for them are

(a) $X = ((-1)^n)$ and $Y = (-(-1)^n)$. Then $X + Y = (0)$, which is convergent as a constant sequence. On the other hand, assume that X is convergent, say $\lim((-1)^n) = L$. Take $\varepsilon = \frac{1}{2} > 0$, then there exists $K \in \mathbb{N}$, such that for all $n > K$, we have

$$|(-1)^n - L| < \frac{1}{2}$$

In particular, we compute for $n > K$

$$|(-1)^n - (-1)^{n+1}| = 2 \text{ and } |(-1)^n - (-1)^{n+1}| \leqslant |(-1)^n - L| + |(-1)^{n+1} - L| < 1$$

i.e. $2 < 1$, which is obviously a contradiction. Thus X can not be convergent.

(b) $X = \left(\frac{1}{n}\right)$ and $Y = (n)$. Then $XY = (1)$, which is convergent as a constant sequence. We have shown that $\lim\left(\frac{1}{n}\right) = 0$, i.e. X is convergent. But Y is not bounded, which implies that it cannot be convergent.

22. (**Page 28, #2**) Suppose that (x_n) and (y_n) are convergent sequences. Prove that $(x_n y_n)$ is convergent as well, and $\lim(x_n y_n) = \lim(x_n)\lim(y_n)$.

Solution: Since both sequences are convergent, they are bounded. In particular, there is $M > 0$ such that $|x_n| < M$ and $|y_n| < M$ for all $n \in \mathbb{N}$. Let $a = \lim(a_n)$ and $b = \lim(b_n)$, then $|a| \leqslant M$ and $|b| \leqslant M$. For any $\varepsilon > 0$, there exist

$K_1 \in \mathbb{N}$ such that $\forall n > K_1$, $|x_n - a| < \frac{1}{2M}\varepsilon$
and $K_2 \in \mathbb{N}$ such that $\forall n > K_2$, $|y_n - b| < \frac{1}{2M}\varepsilon$

We then compute for all $n > K = \max\{K_1, K_2\}$:

$$|x_n y_n - ab| = |x_n(y_n - b) + (x_n - a)b| \leqslant |x_n| \cdot |y_n - b| + |x_n - a| \cdot |b| < M\frac{\varepsilon}{2M} + \frac{\varepsilon}{2M}M = \varepsilon$$

By definition of limits, we see that $\lim(x_n y_n) = ab = \lim(x_n)\lim(y_n)$.

23. (**Page 28, #3**) If $1 > a > 0$, $1 > b > 0$, prove that $\lim(n - \sqrt{(n-a)(n-b)}) = \dfrac{a+b}{2}$.

Solution: We compute using limit rules

$$\lim(n - \sqrt{(n-a)(n-b)}) = \lim\left(\frac{n^2 - (n^2 - (a+b)n + ab)}{n + \sqrt{(n-a)(n-b)}}\right) = \lim\left(\frac{(a+b)n - ab}{n + \sqrt{(n-a)(n-b)}}\right)$$

$$= \lim\left(\frac{a + b + \frac{ab}{n}}{1 + \sqrt{(1 - \frac{a}{n})(1 - \frac{b}{n})}}\right) = \frac{a + b + \lim(\frac{ab}{n})}{1 + \sqrt{(1 + \lim(\frac{a}{n})(1 + \lim(\frac{b}{n})))}} = \frac{a+b}{2}$$

24. (**Page 28, #4**) Give an example of a divergent sequence (x_n) of positive numbers with $\lim(\sqrt[n]{x_n}) = 1$. (Thus, this property cannot be used as a test for convergence.)

Solution: We claim that $X = (n)$ satisfies the requirement. Since X is unbounded, it is divergent. Thus we only have to show that $\lim(\sqrt[n]{n}) = 1$.

To prove the limit, we need to estimate $|\sqrt[n]{n} - 1|$, and compare it with any $\varepsilon > 0$.

First of all, for $n > 1$, we must have $\sqrt[n]{n} > 1$. Thus, when $n > 1$, we may write $\sqrt[n]{n} = 1 + k_n$, for some $k_n > 0$. This then gives by binomial expansion (again when $n > 1$)

$$n = (1 + k_n)^n = 1 + nk_n + \frac{1}{2}n(n-1)k_n^2 + \ldots + k_n^n > 1 + \frac{1}{2}n(n-1)k_n^2$$

It follows that

$$n - 1 > \frac{1}{2}n(n-1)k_n^2 \Longrightarrow 0 < \sqrt[n]{n} - 1 = k_n < \sqrt{\frac{2}{n}}$$

We now verify the definition:

$$\forall \varepsilon > 0, \textit{ find } K \in \mathbb{N}, \text{ such that } \forall n > K, |\sqrt[n]{n} - 1| < \varepsilon$$

Start with $|\sqrt[n]{n} - 1| < \varepsilon$, the computation above gives us

$$|\sqrt[n]{n} - 1| < \varepsilon \iff k_n < \varepsilon \Longleftarrow \sqrt{\frac{2}{n}} < \varepsilon \Longleftarrow \frac{2}{n} < \varepsilon^2 \Longleftarrow n > \frac{2}{\varepsilon^2}$$

Thus, let $K = \left\lfloor \dfrac{2}{\varepsilon^2} \right\rfloor + 1$, then for all $n > K$, we see

$$n > \frac{2}{\varepsilon^2} \Longrightarrow |\sqrt[n]{n} - 1| < \varepsilon$$

which, by definition, gives $\lim(\sqrt[n]{n}) = 1$.

Alternate solution: Consider the sequence $\left(x_n = \left(1 + \frac{1}{\sqrt{n}}\right)^n\right)$. It is straightforward to verify that $\lim(\sqrt[n]{x_n}) = 1$. On the other hand the sequence $\left(y_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\right)$ has a subsequence $\left(\left(1 + \frac{1}{k}\right)^k\right)$ that has limit e. Thus the sequence $\left(x_n = y_n^{\sqrt{n}}\right)$ has a subsequence $(y_{k^2}^k)$ that is unbounded – which implies that (x_n) is unbounded, and thus diverges.

Another alternate solution: The sequence can also be simply chosen as $(x_n = 3 + (-1)^n)$, then there are two subsequences consisting of the even and odd terms respectively that are constant subsequences: one equals 4, and the other 2. Thus this sequence (x_n) diverges. Then $(\sqrt[n]{x_n})$ becomes

$$x_{2k} = \sqrt[2k]{4} = \sqrt[k]{2} \text{ and } x_{2k-1} = \sqrt[2k-1]{2}$$

The sequence $(y_n = \sqrt[n]{2})$ is decreasing and bounded by 1 from below. Thus (y_n) converges by monotone convergence. Let the limit be L. Since $y_{2n}^2 = y_n \Longrightarrow L^2 = L \Longrightarrow L = 0$ or 1. Since $y_n \geqslant 1$, we see that $L = 1$.

The sequence (x_n) is a union of subsequences of (y_n), which converges to 1. A short $\varepsilon - K$ proof using this fact shows that (x_n) in fact converges to 1 as well. (You should make sure you can produce this proof.)

25. (**Page 28, #5**) Suppose that (x_n) is a convergent sequence and (y_n) is such that for any $\varepsilon > 0$ there exists M such that $|x_n - y_n| < \varepsilon$ for all $n \geqslant M$. Does it follow that (y_n) is convergent?

Solution: Yes. The proof goes as following.

Suppose that $\lim(x_n) = L$, then

$$\forall \varepsilon > 0, \exists K_1 \in \mathbb{N}, \text{ such that } \forall n > K_1, |x_n - L| \leqslant \frac{\varepsilon}{2}.$$

Let $K_2 \in \mathbb{N}$ such that for $n > K_2$, $|x_n - y_n| < \frac{\varepsilon}{2}$. Then let $K = \max\{K_1, K_2\}$. For all $n > K$, we have $n > K_1$ and $n > K_2$, which implies that

$$|y_n - L| \leqslant |y_n - x_n| + |x_n - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

By definition, it follows that $\lim(y_n) = L$. In particular, (y_n) is convergent.

26. (**Page 28, #6**) Suppose that (x_n) is a divergent sequence and (y_n) is such that for any $\varepsilon > 0$ there exists M such that $|x_n - y_n| < \varepsilon$ for all $n \geqslant M$. Does it follow that (y_n) is divergent?

Solution: Yes. The proof is by contradiction. Assume that (y_n) is convergent, then the proof of the previous problem shows that (x_n) has to be convergent as well. Contradiction. Thus (y_n) is divergent. **Note:** If this problem is on its own, the proof should include the proof of the previous problem in order to be complete.

27. (**Page 28, #7**) Let $x_1 > 2$ and $x_{n+1} = 3 - \frac{1}{x_n}$ for $n \in \mathbb{N}$. Prove that (x_n) is bounded and monotone, and find the limit.

Solution: First, suppose that $x_2 = x_1$, then we see that (x_n) is a constant sequence, which obviously is bounded and monotone. Let $x = x_1$ then we compute that

$$x = 3 - \frac{1}{x} \Longrightarrow x^2 - 3x + 1 = 0 \Longrightarrow x = \frac{3 \pm \sqrt{5}}{2} \Longrightarrow x = \frac{3 + \sqrt{5}}{2}$$

because $x > 2$.

Next, suppose that $x_2 \neq x_1$. Since $x_1 > 2$, we see that $x_2 = 3 - \frac{1}{x_1} > 3 - \frac{1}{2} > 2$ and $x_2 < 3$.

Assume that $2 < x_k < 3$ for $1 < k \in \mathbb{N}$.

For $n = k + 1$, we compute

$$1 > \frac{1}{x_k} > \frac{1}{3} \Longrightarrow x_{k+1} = 3 - \frac{1}{x_k} > 3 - \frac{1}{2} > 2 \text{ and } x_{k+1} < 3 - \frac{1}{3} < 3$$

Thus (x_n) is bounded below by 2 and bounded above by $\max\{x_1, 3\}$.

To show that (x_n) is monotone, we prove that $x_{n+1} - x_n$ has the same sign as $x_2 - x_1$ for all $n \in \mathbb{N}$. The base case is $n = 1$, which is automatically true.

Assume that $x_{k+1} - x_k$ has the same sign as $x_2 - x_1$.

We compute

$$x_{k+2} - x_{k+1} = \frac{1}{x_k} - \frac{1}{x_{k+1}} = \frac{x_{k+1} - x_k}{x_{k+1}x_k}$$

Since $x_{k+1}x_k > 0$, we see that $x_{k+2} - x_{k+1}$ has the same sign as $x_{k+1} - x_k$, which has the same sign as $x_2 - x_1$.

By Monotone Convergence Theorem, (x_n) converges. Let $L = \lim(x_n)$, then we have

$$L = 3 - \frac{1}{L} \Longrightarrow L^2 - 3L + 1 = 0 \Longrightarrow L = \frac{3 \pm \sqrt{5}}{2}$$

Since $x_n > 2$ for all n, we see that $L \geqslant 2$, thus $L = \frac{3 + \sqrt{5}}{2}$.

28. (**Page 28, #8**) Let (a_n) be an increasing sequence, (b_n) be a decreasing sequence, and assume that $a_n \leqslant b_n$ for all $n \in \mathbb{N}$. Prove that $\lim(a_n) \leqslant \lim(b_n)$. Use this to prove the Nested Interval Property 3.50 from the Monotone Convergence Theorem 4.29.

Solution: Since $a_n \leqslant b_n$ for all $n \in \mathbb{N}$ and b_n is decreasing, we see that $a_n \leqslant b_n \leqslant b_1$ for all $n \in \mathbb{N}$. Thus (a_n) is an increasing sequence bounded above. Thus (a_n) converges by Monotone Convergence Theorem. Let $a = \lim(a_n)$.

Similarly, since $b_n \geqslant a_n \geqslant a_1$ for all $n \in \mathbb{N}$, we see that (b_n) is a decreasing sequence bounded below. This implies that (b_n) converges by Monotone Convergence Theorem. Let $b = \lim(b_n)$.

For any $m \in \mathbb{N}$, consider the m-tail (a_{m+k}) of (a_n), then we have $a_{m+k} \leqslant b_{m+k} \leqslant b_m$ since (b_n) is decreasing. It follows that for any $m \in \mathbb{N}$, we have $a = \lim(a_n) = \lim(a_{m+k}) \leqslant b_m$. From this, we see that $a \leqslant \lim(b_m) = b$.

Let now $I_n = [a_n, b_n]$ be a nested sequence of closed intervals, then (a_n) is an increasing sequence and (b_n) is a decreasing sequence, and indeed $a_n \leqslant b_n$ for all $n \in \mathbb{N}$. Thus we may apply the above arguments and see that $a := \lim(a_n) < \lim(b_n) =: b$.

We claim that $a \in \bigcap_{n\in\mathbb{N}} I_n$. We have proven that $a \leqslant b_m$ for all $m \in \mathbb{N}$. Thus we only need to show that $a \geqslant a_m$ for all $m \in \mathbb{N}$. Since (a_n) is increasing, we see that for any $m \in \mathbb{N}$, the m-tail (a_{m+k}) of (a_n) satisfies $a_{m+k} \geqslant a_m$. Thus $a = \lim(a_n) = \lim(a_{m+k}) \geqslant a_m$ for all $m \in \mathbb{N}$. It follows that $a \in I_m$ for all $m \in \mathbb{N}$.

29. (**Page 28, #9**) Let A be an infinite subset of $\mathbb{R}$ that is bounded above and let $u := \sup(A)$. Prove that there exists an increasing sequence (x_n) with $x_n \in A$ for all $n \in \mathbb{N}$ such that $u = \lim(x_n)$.

Solution: Note that (x_n) is increasing means that $x_{n+1} \geqslant x_n$ for all $n \in \mathbb{N}$.

There are two cases. Suppose that $u \in A$, then the constant sequence $(x_n = u)$ is what we need.

Now suppose that $u \notin A$, it follows that $\forall y \in A$, $y < u$. Since $u = \sup(A)$, for any $\varepsilon > 0$, there exists $x \in A$ such that $x > u - \varepsilon$. We now construct the sequence (x_n) as following.

For $\varepsilon_1 = 1$, let $x_1 \in A$ be an element that $x_1 > u - 1$.

For $\varepsilon_2 = \min\left\{\dfrac{1}{2}, u - x_1\right\}$, since $u - x_1 > 0$, we see that $\varepsilon_2 > 0$. Let $x_2 \in A$ be an element that $x_2 > u - \varepsilon_2$. Here we see that $x_2 > u - \varepsilon_2 \geqslant u - (u - x_1) = x_1$.

For $1 < k \in \mathbb{N}$, we define $x_{k+1} \in A$ inductively as following. Let $\varepsilon_{k+1} = \min\left\{\dfrac{1}{k+1}, u - x_k\right\}$. Since $u - x_k > 0$, we see that $\varepsilon_{k+1} > 0$. Let $x_{k+1} \in A$ be an element that $x_{k+1} > u - \varepsilon_{k+1}$. We also see that

$$x_{k+1} > u - \varepsilon_{k+1} \geqslant u - (u - x_k) = x_k$$

By mathematical induction, we have now a sequence (x_n) in A, which is increasing, and for any $\varepsilon > 0$, let $K = \left\lfloor \dfrac{1}{\varepsilon} \right\rfloor + 1$, when $n > K$, it implies that

$$n > \frac{1}{\varepsilon} \Longrightarrow \varepsilon > \frac{1}{n} \geqslant \varepsilon_n > u - x_n = |x_n - u|$$

By definition, we see that $\lim(x_n) = u$.

30. (**Page 32, #1**) Let (x_n) be defined by $x_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)}$. Prove that (x_n) is convergent and find its limit.

Solution: It's obvious that $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. Since $x_n > 0$ as well for all $n \in \mathbb{N}$, monotone convergence implies that (x_n) is convergent. The claim is that the $\lim(x_n) = 0$.

We prove by contradiction. Assume that $L = \lim(x_n) \neq 0$. Since $x_n > 0$, we see that we must have $L > 0$. In particular, let $y_n = \dfrac{1}{x_n}$ for all $n > 0$, we see that $\lim(y_n) = \dfrac{1}{L} > 0$. We write out

$$y_n = \frac{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)} = \frac{2}{3} \cdot \frac{4}{5} \cdot \ldots \cdot \frac{2n-2}{2n-1} \cdot 2n$$

Since for each $k > 1$, we have

$$k^2 - 1 < k^2 \Longrightarrow \frac{k-1}{k} < \frac{k}{k+1}$$

it follows that

$$0 < x_n = \frac{1}{2} \cdot \frac{3}{4} \cdot \dots \cdot \frac{2n-1}{2n} < \frac{2}{3} \cdot \frac{4}{5} \cdot \dots \cdot \frac{2n}{2n+1} = \frac{1}{2n+1} y_n$$

Since $\lim\left(\frac{1}{2n+1} y_n\right) = \lim\left(\frac{1}{2n+1}\right) \lim(y_n) = 0 \times \frac{1}{L} = 0$, the squeeze theorem implies that $L = \lim(x_n) = 0$. This is in contradiction to the assumption that $L \neq 0$.

Thus the assumption is false and $\lim(x_n) = 0$.

Alternate solution: Similarly we see that (x_n) converges by monotone convergence theorem. Let $L = \lim(x_n)$ and we compute

$$x_n^2 = \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot \dots \cdot (2n-1)^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot \dots \cdot (2n)^2} < \frac{1 \cdot 3^2 \cdot 5^2 \cdot \dots \cdot (2n-1)^2}{(2^2-1) \cdot (4^2-1) \cdot (6^2-1) \cdot \dots \cdot ((2n)^2-1)}$$

$$= \frac{1 \cdot 3^2 \cdot 5^2 \cdot \dots \cdot (2n-1)^2}{(1 \cdot 3) \cdot (3 \cdot 5) \cdot (5 \cdot 7) \cdot \dots \cdot ((2n-1)(2n+1))} = \frac{1}{2n+1}$$

Since $\lim\left(\frac{1}{2n+1}\right) = 0$, squeeze theorem implies that $L^2 = \lim(x_n^2) = 0$. Thus $L = 0$.

31. (**Page 32, #2**) Let (x_n) be defined by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$. Prove that (x_n) is convergent and find its limit.

Solution: We prove that (x_n) is increasing and bounded above by 2.

For $n = 1$, we have $x_1 = \sqrt{2} < 2$ and $x_2 = \sqrt{2 + \sqrt{2}} > \sqrt{2} = x_1$.

Assume that we have $x_k < 2$ and $x_{k+1} > x_k$ for some $k \in \mathbb{N}$.

Then we see $x_{k+1} = \sqrt{2 + x_k} < \sqrt{2 + 2} = 2$. Furthermore, we compute

$$x_{k+2} - x_{k+1} = \sqrt{2 + x_{k+1}} - \sqrt{2 + x_k} = \frac{x_{k+1} - x_k}{\sqrt{2 + x_{k+1}} + \sqrt{2 + x_k}} > 0$$

Thus $x_{k+2} > x_{k+1}$.

By mathematical induction, we see that (x_n) is increasing and bounded above by 2.

It follows by monotone convergence theorem that (x_n) converges. Let $L = \lim(x_n)$. Taking limit on both sides of the recursive definition of (x_n), we get

$$L = \sqrt{2 + L} \iff L^2 - L - 2 = 0 \iff (L-2)(L+1) = 0 \iff L = 2 \text{ or } L = -1$$

Since $x_n > 0$ for all $n \in \mathbb{N}$, we must have $L \geqslant 0$. Thus $L = 2$.

32. (**Page 32, #3**) Let $0 < a_1 < b_1$ and define (a_n) and (b_n) by

$$a_{n+1} = \sqrt{a_n b_n} \text{ and } b_{n+1} = \frac{1}{2}(a_n + b_n)$$

Prove that (a_n) and (b_n) converges and they have the same limit.

Solution: We show that (a_n) is increasing and (b_n) is decreasing, and always $0 < a_n < b_n$.

When $n = 1$, we have $0 < a_1 < b_1$ by definition, and

$$a_1 = \sqrt{a_1 \cdot a_1} < \sqrt{a_1 b_1} = a_2 \text{ and } b_1 = \frac{1}{2}(b_1 + b_1) > \frac{1}{2}(a_1 + b_1) = b_2$$

Assume that for $n = k$, the statement holds, i.e. $0 < a_k < b_k$,

$$a_k < a_{k+1} \text{ and } b_k > b_{k+1}$$

For $n = k + 1$, by arithmetic-geometric inequality, since $a_k \neq b_k$, we get

$$0 < a_{k+1} = \sqrt{a_k b_k} < \frac{1}{2}(a_k + b_k) = b_{k+1}$$

Then we get

$$a_{k+1} = \sqrt{a_{k+1} \cdot a_{k+1}} < \sqrt{a_{k+1} b_{k+1}} = a_{k+2} \text{ and } b_{k+1} = \frac{1}{2}(b_{k+1} + b_{k+1}) > \frac{1}{2}(a_{k+1} + b_{k+1}) = b_{k+2}$$

By mathematical induction, we see that (a_n) is increasing and (b_n) is decreasing, and always $0 < a_n < b_n$. By monotone convergence theorem, we see that both (a_n) and (b_n) are convergent. Let $A = \lim(a_n)$ and $B = \lim(b_n)$. The recursive equation defining (b_n) induces an identity involving the limits:

$$B = \frac{1}{2}(A + B) \Longrightarrow A = B$$

33. (**Page 32, #4**) Let $X = (x_n)$ and $Y = (y_n)$ be given sequences, and let the sequence $Z = (z_n)$ be defined by $z_1 := x_1$, $z_2 := y_1$, $\ldots$, $z_{2n-1} := x_n$, $z_{2n} := y_n$, $\ldots$. Show that Z is convergent if and only if both X and Y are convergent and $\lim X = \lim Y$.

Solution: "$\Longrightarrow$:" Since Z is convergent, it follows that all its subsequence are convergent and have the same limits. In particular, the subsequences $(z_{2k-1} = x_k)$ and $(z_{2k} = y_k)$ are convergent and have the same limits.

"$\Longleftarrow$:" Since X and Y are convergent and has the same limit, let $L = \lim X = \lim Y$. Then

$$\forall \varepsilon > 0, \exists M \in \mathbb{N} \text{ such that } \forall n > M, |x_n - L| < \varepsilon \text{ and } |y_n - L| < \varepsilon.$$

It follows that for the same $\varepsilon > 0$, let $K = 2M$, then for $n > 2M$, we see that if $n = 2k - 1$ is odd, we have $k > M$ and $z_n = x_k$, which implies that $|z_n - L| = |x_k - L| < \varepsilon$; while if $n = 2k$ is even, we have $k > M$ and $z_n = y_k$, which implies that $|z_n - L| = |y_k - L| < \varepsilon$.

Thus we verified the definition of $\lim Z = L$, i.e. Z is convergent.

34. (**Page 32, #5**) Suppose that $x_n \geqslant 0$ for all $n \in \mathbb{N}$ and that $\lim((-1)^n x_n)$ exists. Prove that (x_n) converges and find its limit.

Solution: Suppose that $\lim((-1)^n x_n) = L$, then it implies that the two subsequences (x_{2k}) and $(-x_{2k-1})$ both converges and $L = \lim(x_{2k}) = \lim(-x_{2k-1})$. Since $x_n \geqslant 0$ for all $n \in \mathbb{N}$, we see in particular

$$x_{2k} > 0 \Longrightarrow L \geqslant 0 \text{ and } -x_{2k-1} < 0 \Longrightarrow L \leqslant 0$$

Thus $L = 0$, and $\lim(x_{2k}) = \lim(x_{2k-1}) = 0$. From the proof of Problem 22 in this assignment, we see that this implies that (x_n) is convergent and $\lim(x_n) = 0$.

35. (**Page 32, #6**) Let (x_n) be defined by $x_1 = 1$ and $x_n = x_{n-1} + \frac{1}{n^2}$ for all $n > 1$. Prove that (x_n) is convergent.

Solution: Obviously (x_n) is increasing. We prove that it converges by showing that it is bounded above. For any $n \in \mathbb{N}$, we may find $k \in \mathbb{N}$ such that $2^{k-1} \leqslant n < 2^k$. Thus we may estimate x_n as following:

$$x_n = \sum_{i=1}^{n} \frac{1}{i^2} \leqslant \sum_{i=1}^{2^k-1} \frac{1}{i^2} = \sum_{j=1}^{k} \sum_{i=2^{j-1}}^{2^j-1} \frac{1}{i^2} < \sum_{j=1}^{k} \frac{2^j - 2^{j-1}}{2^{2(j-1)}} = \sum_{j=1}^{k} \frac{1}{2^{j-1}} < 2$$

Thus by monotone convergence theorem, (x_n) is convergent.

Alternate solution: We can prove the boundedness by noticing that for $n > 1$:

$$\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$$

which implies that

$$x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 1 + \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \ldots + \frac{1}{n-1} - \frac{1}{n} = 2 - \frac{1}{n} < 2$$

Thus (x_n) is bounded above and again by monotone convergence theoremm, (x_n) is convergent.

36. (**Page 32, #7**) Determine if the sequence (y_n) as defined in the following is convergent or divergent:

$$y_n := \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n} \text{ for } n \in \mathbb{N}$$

Solution: First, we show that (y_n) is bounded. Obviously $y_n > 0$, for all $n \in \mathbb{N}$. We compute

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n} < \underbrace{\frac{1}{n} + \ldots + \frac{1}{n}}_{2n} = 2n \times \frac{1}{n} = 2$$

Thus (y_n) is bounded below by 0 and above by 2.

Next, we show that (y_n) is monotone. For $n \in \mathbb{N}$, we compute

$$y_{n+1} - y_n = \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{3(n+1)} - \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n}\right)$$

$$= \frac{1}{3n+1} + \frac{1}{3n+2} + \frac{1}{3(n+1)} - \frac{1}{n+1} > 3 \times \frac{1}{3(n+1)} - \frac{1}{n+1} = 0$$

Thus $y_{n+1} > y_n$ for all $n \in \mathbb{N}$, i.e. (y_n) is increasing.

By monotone convergence theorem, we see that (y_n) converges.

37. (**Page 32, #8**) Let (x_n) be a sequence and define a new sequence (a_n) where $a_n = \frac{1}{n}\sum_{i=1}^{n} x_i$.

Prove that (x_n) is convergent $\Longrightarrow$ (a_n) is convergent. Is the opposite implication also true? Fully justify your conclusion.

Solution: Since (x_n) is convergent, let $\lim(x_n) = L$, then

$\forall \varepsilon > 0$, then $\exists K \in \mathbb{N}$ such that $\forall n > K$, $|x_n - L| < \frac{\varepsilon}{2}$.

Moreover, convergence also implies that the sequence (x_n) is bounded, i.e. there is $M > 0$ such that $|x_n| < M$ for all $n \in \mathbb{N}$. We then compute for (a_n) we have

$$|a_n - L| = \left|\frac{1}{n}\sum_{i=1}^{n} x_n - L\right| = \left|\frac{1}{n}\sum_{i=1}^{n}(x_n - L)\right| \leqslant \frac{1}{n}\sum_{i=1}^{n}|x_n - L|$$

In particular, for $n > M$, we have

$$|a_n - L| \leqslant \frac{1}{n}\sum_{i=1}^{M}|x_n - L| + \frac{1}{n}\sum_{i=M+1}^{n}|x_n - L| < \frac{1}{n}M(P + |L|) + \frac{1}{n}(n - M)\frac{\varepsilon}{2}$$

Let $K' = \left\lfloor \max\left(M, \frac{2K(M + |L|)}{\varepsilon}\right)\right\rfloor + 1$, then for $n > K'$ we have

$$|a_n - L| < \frac{1}{n}K(M + |L|) + \frac{1}{n}(n - M)\frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, by definition of limit, we see that (a_n) converges. In fact it has the same limit as (x_n).

The opposite implication fails. In fact, when the sequence (x_n) is alternating, say $x_n = (-1)^n$, we have

$$a_{2k} = 0 \text{ and } a_{2k-1} = -\frac{1}{k} \text{ for all } k \in \mathbb{N}$$

It follows that for any $\varepsilon > 0$, we may take $K = \left\lfloor \frac{2}{\varepsilon} \right\rfloor + 2$, then $\forall n > K$, we have

$$n = 2k \Longrightarrow |a_n - 0| = \left|\frac{1}{k} - 0\right| = \frac{2}{n} < \varepsilon, \text{ while } n = 2k - 1 \Longrightarrow |a_n - 0| = 0 < \varepsilon$$

Thus $\lim(a_n) = 0$. On the other hand, $((-1)^n)$ is not convergent.