13. Taylor & Maclaurin Series

- ◇ We've seen that a power series $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ may represent a function $f(x)$ on its interval of convergence, but is this representation unique?
- ◇ Are the coefficients, the c_j's, special in some way?
  That is, are the c_j's predetermined by $f(x)$?
- ◇ The answer to these questions is 'yes' and here's why:

TAYLOR & MACLAURIN SERIES: A NEW WAY TO CONSIDER POWER SERIES

Suppose we have a power series representation for $f(x)$ on its interval of convergence centred at a:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$$

What is $f(a)$? $f(a) = c_0 + c_1(a-a) + c_2(a-a)^2 + \ldots = c_0$

What is $f'(a)$? Well, $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \ldots$

$\Rightarrow f'(a) = c_1$

What is $f''(a)$? Well, $f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + 4\cdot 5\cdot c_5(x-a)^3 + \ldots$

$\Rightarrow f''(a) = 2c_2 \qquad \Rightarrow c_2 = \dfrac{f''(a)}{2}$

What is $f'''(a)$? Well, $f'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + 3\cdot 4\cdot 5c_5(x-a)^2 + \ldots$

$\Rightarrow f'''(a) = 2\cdot 3c_3 \qquad \Rightarrow c_3 = \dfrac{f'''(a)}{2\cdot 3}$

In general, we see the pattern:

$$f^{(n)}(a) = 2\cdot 3\cdot \cdots (n-2)(n-1)(n)c_n \qquad \Rightarrow \boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

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* These notes are solely for the personal use of students registered in MAT1322.

Thus, the coefficients of the power series representation are determined by the function $f(x)$ and the centre a, hence they are unique to f.

What we get is

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

- ▷ Thinking of the power series representation for $f(x)$ in this way, we call this series the **Taylor series of f centred at a.**

- ▷ In the special case when the centre is $a = 0$, it's called the **Maclaurin series of f**

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n}$$

Example 13.1. What is the Maclaurin series for $f(x) = \dfrac{1}{1-x}$?

We already Know the power series representation for $\frac{1}{1-x}$ centred at 0:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (\text{for } |x|<1)$$

↑ this is the Maclaurin series for $\frac{1}{1-x}$

Check its coefficients in the new light

$c_n = \frac{f^{(n)}(0)}{n!}$ where $f(x) = \frac{1}{1-x}$.

You will see that $f^{(n)}(0) = n!$ for all $n \geq 0$.

$$f(x) = (1-x)^{-1} \Rightarrow f(0) = 1$$
$$f'(x) = -(1-x)^{-2}(-1) = (1-x)^{-2} \Rightarrow f'(0) = 1$$
$$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3} \Rightarrow f''(0) = 2$$
$$f'''(x) = -3\cdot2\cdot(1-x)^{-4}(-1) = 3\cdot2\cdot(1-x)^{-4} \Rightarrow f'''(0) = 3\cdot2$$
$$f^{(4)}(x) = -4\cdot3\cdot2\cdot(1-x)^{-5}(-1) = 4\cdot3\cdot2\cdot(1-x)^{-5} \Rightarrow f^{(4)}(0) = 4\cdot3\cdot2$$
$$\vdots$$

In general, $f^{(n)}(0) = n!$ $\therefore$ $c_n = \frac{f^{(n)}(0)}{n!} = \frac{n!}{n!} = 1$

Example 13.2. Determine the Maclaurin series for $\boxed{f(x) = e^x}$

$$f(x) = e^x \Rightarrow f(0) = 1$$
$$f'(x) = e^x \Rightarrow f'(0) = 1$$
$$f''(x) = e^x \Rightarrow f''(0) = 1$$
$$\vdots$$
$$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1$$

$\therefore$ the Maclaurin series for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Ratio Test $\displaystyle \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \bigg/ \frac{x^n}{n!}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$ for all x

$(R = \infty)$

good for all x

In particular, if $x = 1$, we get another definition for the number e

$$\boxed{e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \cdots}$$

other definitions for e that you may recall:

$$\boxed{e \text{ is the number such that } \lim_{h \to 0} \frac{e^h - 1}{h} = 1}$$

$$\boxed{e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x}$$

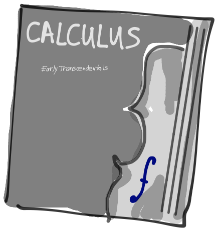
Exercise 13.3. Use a Maclaurin series to estimate $\displaystyle\int_0^1 e^{-x^2}\,dx$ to within an error of 0.001

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}\Bigg]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} \longleftarrow$$

this is an alternating series and it passes the AST.

By Alternating Series Estimation Theorem

$$|\text{error}| = |S - S_n| \le b_{n+1} \implies \text{solve for } n \text{ when } b_{n+1} \le 0.001$$



see Stewart page 768-9 for solution

Example 13.4. Determine the Maclaurin series for $\boxed{f(x) = \sin(x)}$

$f(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos(x) \Rightarrow f'(0) = 1$

$f''(x) = -\sin(x) \Rightarrow f''(0) = 0$

$f'''(x) = -\cos(x) \Rightarrow f'''(0) = -1$

$f^{(4)}(x) = \sin(x) \Rightarrow f^{(4)}(0) = 0$

$\vdots$

↰ this pattern repeats infinitely

$\therefore$ the Maclaurin series for sin(x) is

$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + \dots$$

$$\therefore \quad \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!} \Big/ \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right| = \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)} = 0 < 1 \text{ for all } x \quad (R = \infty)$$

Now, we can differentiate this to get the Maclaurin series for $\boxed{f(x) = \cos(x)}$

$$\cos(x) = \frac{d}{dx}[\sin x] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] = \sum_{n=0}^{\infty} \frac{\cancel{(2n+1)}(-1)^n x^{2n}}{\cancel{(2n+1)!}\,(2n)!}$$

$$\therefore \quad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Example 13.5. How would we estimate $\int_0^1 \cos(x^2)dx$ to within an error of 10^{-7} ?

$$\int_0^1 \cos(x^2)dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = C + \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!(4n+1)}\right]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(4n+1)}$$

We can use the A.S.E.T. (verify this!) We see that $b_5 = \frac{1}{10!\cdot 21} \approx 1.3 \times 10^{-8} < 10^{-7}$

$$\therefore \int_0^1 \cos(x^2)dx \approx 1 - \frac{1}{2!\cdot 5} + \frac{1}{4!\cdot 9} - \frac{1}{6!\cdot 13} + \frac{1}{8!\cdot 17} \approx 0.904524251\dots$$

THE BINOMIAL SERIES

Let's work out the Maclaurin series for $\boxed{f(x) = (1+x)^k}$

$$f(x) = (1+x)^k \implies f(0) = 1$$

$$f'(x) = k(1+x)^{k-1} \implies f'(0) = k$$

$$f''(x) = k(k-1)(1+x)^{k-2} \implies f''(0) = k(k-1)$$

$$f'''(x) = k(k-1)(k-2)(1+x)^{k-3} \implies f'''(0) = k(k-1)(k-2)$$

$$\vdots$$

$$f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1-x)^{k-n} \implies f^{(n)}(0) = k(k-1)\cdots(k-n+1)$$

$\therefore$ the Maclaurin series for $f(x) = (1+x)^k$ is

$$\boxed{(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n}$$

this is called the Binomial Series

For $k \in \mathbb{R}$ and $n \in \mathbb{Z}, n \geq 1$, we denote by $\binom{k}{n}$ the binomial coefficient

$$\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} \quad \text{and} \quad \binom{k}{0} = 1$$

$\implies$ the Binomial Series can be written as $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{k(k-1)\cdots(k-(n+1)+1)x^{n+1}}{(n+1)!} \Big/ \frac{k(k-1)\cdots(k-n+1)x^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{(k-n)}{n+1} x \right| = |x|$$

Thus, the binomial series $(1+x)^k$ converges if $|x| < 1$ and diverges if $|x| > 1$ $\quad (R = 1)$

Example 13.6. Find the Maclaurin series for $\sqrt{1+x}$ or $(1+x)^{1/2}$ so $k = 1/2$

For $n \geqslant 2$, we have

$$\binom{1/2}{n} = \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n+1)}{n!} = \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\cdots(\frac{3}{2}-\frac{2n}{2})}{n!}$$

$$= \frac{(\frac{1}{2})(-1)^{n-1}(\frac{1}{2})^{n-1}(1)(3)(5)\cdots(2n-3)}{n!} = \frac{(-1)^{n-1}(1)(3)(5)\cdots(2n-3)}{2^n n!}$$

Also, $\binom{1/2}{0} = 1$ and $\binom{1/2}{1} = \frac{\frac{1}{2}}{1!} = \frac{1}{2}$

$$\therefore \sqrt{1+x} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^n(1)(3)\cdots(2n-3)}{2^n n!}$$

Example 13.7. Find the Maclaurin series for $\frac{1}{(2+x)^3}$ or $(2+x)^{-3}$ so $k = -3$

We have $(2+x)^{-3} = \left(2(1+\frac{x}{2})\right)^{-3} = 2^{-3}(1+\frac{x}{2})^{-3}$

for $n \geqslant 2$, we have

$$\binom{-3}{n} = \frac{(-3)(-3-1)\cdots(-3-n+1)}{n!} = \frac{(-3)(-4)\cdots(-n-2)}{n!} = (-1)^n \frac{\cancel{3}\cdot\cancel{4}\cdot\cdots\cdot\cancel{n}\cdot(n+1)(n+2)}{1\cdot 2\cdot\cancel{3}\cdot\cancel{4}\cdot\cdots\cdot\cancel{(n)}} = \frac{(-1)^n(n+1)(n+2)}{2}$$

$\binom{-3}{0} = 1$ and $\binom{-3}{1} = -3$

$$\therefore \frac{1}{(2+x)^3} = \frac{1}{8}\left(1+\frac{x}{2}\right)^{-3} = \frac{1}{8}\sum_{n=0}^{\infty}\binom{-3}{n}\left(\frac{x}{2}\right)^n = \frac{1}{8}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)}{2}\frac{x^n}{2^n}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)}{2^{n+4}}x^n$$

THE LIST OF SERIES YOU NEED TO KNOW

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \qquad (R=1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots \qquad (R=\infty)$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad (R=\infty)$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad (R=\infty)$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \qquad (R=1)$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \qquad (R=1)$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \ldots \qquad (R=1)$$

STUDY GUIDE

☐ ↑ **The List of Series You Need to Know** ↑

☐ **Taylor Series for $f(x)$ at a** $\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$

☐ **Maclaurin Series for $f(x)$** $\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n}$

Exer. (Stewart, 8th ed.)	§**11.10** pg. 771:	3, 5, 7, 9, 11, 15, 16, 21, 25, 27, 29, 31, 33, 35, 37, 45, 47, 49, 51, 53, 55, 57, 59, 61, 63, 65, 67