7. The Integral Test & The Comparison Test

- ◇ Determining whether an infinite series $\sum_{n=1}^{\infty} a_n$ converges or diverges amounts to determining the limit of the sequence of its partial sums $\lim_{n\to\infty} s_n$.
- ◇ In order to determine $\lim_{n\to\infty} s_n$, we may need to find an expression for the nth partial sum s_n (*e.g.* telescoping series, geometric series).
- ∘ To find and expression for the nth partial sum can be quite challenging, so we'd like another way to determine whether a series converges or diverges.

THE INTEGRAL TEST

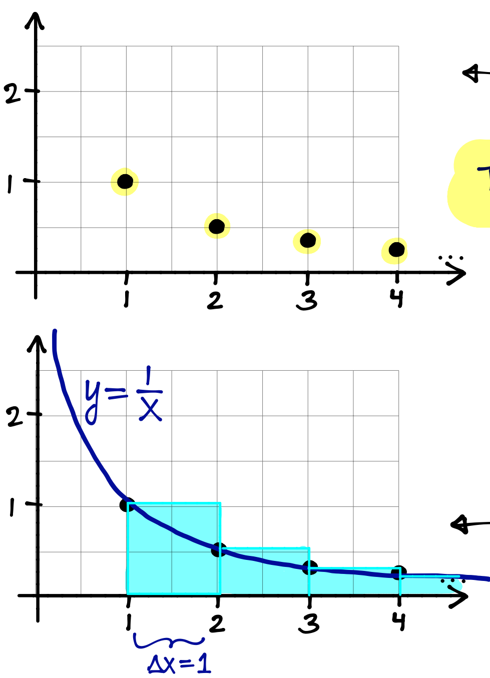
As some of you have already remarked, series seem related to improper integrals.

Note $\sum_{n=1}^{\infty} f(n) \neq \int_1^{\infty} f(x)\,dx$ but they are related...

Example 7.1. the harmonic series:

graph of harmonic sequence

area represents the sum of the harmonic series



← the graph of $f(n) = \frac{1}{n}$ $(n \in \mathbb{Z}^+)$

The harmonic sequence is $\{\frac{1}{n}\} = 1, \frac{1}{2}, \frac{1}{3}, \ldots$

The harmonic series is

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

← the series resembles a Riemann sum approximation of $\int_1^{\infty} \frac{1}{x}\,dx$ using rectangles of height $f(n) = \frac{1}{n}$ width $\Delta x = 1$ and left endpoints

How does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ compare with $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$?

Well, $\sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x}\,dx$ and we already know that $\int_1^{\infty} \frac{1}{x}\,dx$ diverges

$\therefore \sum_{n=1}^{\infty} \frac{1}{n}$ diverges too.

* These notes are solely for the personal use of students registered in MAT1322.

The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$, and let $a_n = f(n)$.

Then

$$\sum_{n=1}^{\infty} a_n \text{ is convergent if and only if } \int_1^{\infty} f(x)\,dx \text{ is convergent.}$$

Thus,

- If $\displaystyle\int_1^{\infty} f(x)dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\displaystyle\int_1^{\infty} f(x)dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Consequences for p series

The p series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} dx$ is $\begin{cases} \text{convergent if } p > 1. \\ \text{divergent if } p \leq 1 \end{cases}$

(just like $\int_1^{\infty} \frac{1}{x^p} dx$)

Example 7.2. Does $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n(\ln n)^2}$ converge? Take $f(x) = \dfrac{2}{x(\ln x)^2}$.

Then $f(x)$ is continuous, positive, and decreasing for $x \geq 2$ ✓

Now, $$\int_2^{\infty} \frac{2}{x(\ln x)^2} dx = \lim_{t\to\infty} \int_2^t \frac{2}{x(\ln x)^2} dx$$

$$= \lim_{t\to\infty} \left[-\frac{2}{\ln x}\right]_2^t$$

$$= \lim_{t\to\infty} -\frac{2}{\ln t} - \left(-\frac{2}{\ln 2}\right)$$

$$= 0 + \frac{2}{\ln 2}$$

u-substitution
$u = \ln x$
$\frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$
$\Rightarrow \int \frac{2}{x(\ln x)^2} dx = 2\int u^{-2} du$
$= 2(-u^{-1})$
$= -2(\ln x)^{-1}$

$\therefore \displaystyle\int_2^{\infty} \frac{2}{x(\ln x)^2} dx$ converges. By the Integral Test, so too does $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n(\ln n)^2}$

Example 7.3. Does $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converge?

Let $f(x) = \frac{x}{x^2+1}$. Then $f(x)$ is positive and continuous on $[1, \infty)$.

Is $f(x)$ decreasing?

Check: $f'(x) = \frac{(1)(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} < 0$ for $x > 1$

$\therefore$ yes, $f(x)$ is decreasing

Now, $\displaystyle\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{x}{x^2+1}\,dx$

u-sub. $u = x^2+1$
$\Rightarrow du = 2x\,dx$
$\Rightarrow dx = \frac{du}{2x}$
$\Rightarrow \int \frac{x}{x^2+1}dx = \int \frac{1}{u}\cdot\frac{du}{2}$

$$= \lim_{t\to\infty} \tfrac{1}{2}\ln(x^2+1)\Big]_1^t$$

$$= \lim_{t\to\infty} \tfrac{1}{2}\ln(t^2+1) - \tfrac{1}{2}\ln(1^2+1)$$

$$= \infty - \tfrac{1}{2}\ln 2$$

$$= \infty$$

$\therefore \displaystyle\int_1^\infty \frac{x}{x^2+1}\,dx$ diverges. By the Integral Test, so too does $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^2+1}$

Alternatively, we might have observed that

$$\frac{x}{x^2+1} \geq \frac{x}{x^2+x^2} = \frac{1}{2}\left(\frac{1}{x}\right) \text{ for all } x \geq 1$$

Thus, $\displaystyle\frac{1}{2}\int_1^\infty \frac{1}{x}dx \leq \int_1^\infty \frac{x}{x^2+1}dx$

↑since $\displaystyle\int_1^\infty \frac{1}{x}dx$ diverges, so too does $\displaystyle\int_1^\infty \frac{x}{x^2+x}dx$

Example 7.4. Does $\sum_{n=1}^{\infty} \frac{3}{n^2+7n+1}$ converge?

First, $\frac{3}{n^2+7n+1}$ is roughly like $\frac{1}{n^2}$ so we suspect it converges...

Thus, we're looking for an upper bound like $\frac{1}{n^2}$ (or some constant multiple of)

Well, $\frac{3}{n^2+7n+1} < \frac{3}{n^2} = 3 \cdot \frac{1}{n^2}$ for all $n \geq 1$,

and we know the p series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges ($p=2>1$).

So

$$0 \leq \sum_{n=1}^{\infty} \frac{3}{n^2+7n+1} < \sum_{n=1}^{\infty} \frac{3}{n^2} = 3 \sum_{n=1}^{\infty} \frac{1}{n^2} \implies \sum_{n=1}^{\infty} \frac{3}{n^2+7n+1} \text{ converges too.}$$

The Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is convergent. (or for all $n \geq n_0$)

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent. (or for all $n \geq n_0$)

Example 7.5. Does $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ converge?

For all $k \geq 3$, we have $0 < \frac{1}{k} < \frac{\ln k}{k}$ because $1 < \ln k \iff e < k$

↑ it's okay that finitely many terms do not satisfy the inequality...

Since the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, so too will $\sum_{k=1}^{\infty} \frac{\ln k}{k}$

Thus, by the Comparison Test, $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ also diverges.

STUDY GUIDE

☐ **The Integral Test** ☐ p **series** ☐ **The Comparison Test**

Exercises	§**11.3** pg. 725:	3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 29
(Stewart, 8th ed.)	§**11.4** pg. 731:	1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 37