

10. Ratio & Root Tests & General Strategies for Testing Series

Series Methods/Tests so far:

- Find explicit formula for n th partial sum S_n . Then $\sum a_k = \lim_{n \rightarrow \infty} S_n$.
- Test for Divergence: if $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum a_n$ is divergent.
- If $\sum a_n$ is a geometric series with common ratio r , then it converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.
- Integral Test: if $f(x)$ is a continuous, positive and (eventually) decreasing function on the infinite interval of interest, then the series $\sum_{n=a}^{\infty} f(n)$ is convergent if and only if the improper integral $\int_a^{\infty} f(x)dx$ is convergent.
- p -series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if $p \leq 1$.
- Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $0 \leq \sum b_n \leq \sum a_n$ and we know $\sum a_n$ is convergent, then $\sum b_n$ must also be convergent. If $0 \leq \sum b_n \leq \sum a_n$ and we know $\sum b_n$ is divergent, then $\sum a_n$ must also be divergent.
- Limit Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$ for some positive real number $c > 0$, then $\sum a_n$ and $\sum b_n$ are either both convergent or both divergent.
- AST: Suppose $\sum a_n$ is an alternating series and $b_n = |a_n|$. If $b_{n+1} \leq b_n$ for all $n \geq n_0$ and $\lim_{n \rightarrow \infty} b_n = 0$, then $\sum a_n$ is convergent.
- Absolute convergence: if $\sum |a_n|$ is convergent, then $\sum a_n$ must also be convergent.

THE RATIO TEST

- i. If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ and $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- ii. If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ and $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
- iii. If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, then the Ratio Test is inconclusive.

Example 10.1. $\sum_{n=1}^{\infty} \frac{10^n}{n!}$

Let $n \in \mathbb{Z}, n \geq 0$. Then "n factorial", denoted $n!$, is defined recursively as follows: $0! = 1$

for $n \geq 1, n! = n(n-1)!$

hence $n! = n(n-1)(n-2)\cdots(2)(1)$

For this series, $a_n = \frac{10^n}{n!}$ and $a_{n+1} = \frac{10^{n+1}}{(n+1)!}$

$$\begin{aligned} \text{Ratio Test: } \lim_{n \rightarrow \infty} \left| \frac{10^{n+1}/(n+1)!}{10^n/n!} \right| &= \lim_{n \rightarrow \infty} \frac{10^{n+1} n!}{10^n (n+1)!} \\ &= \lim_{n \rightarrow \infty} \frac{10 \cancel{n!}}{(n+1) \cancel{n!}} \\ &= \lim_{n \rightarrow \infty} \frac{10}{n+1} \\ &= 0 < 1 \end{aligned}$$

$\therefore \sum_{n=1}^{\infty} \frac{10^n}{n!}$ is absolutely convergent by virtue of the Ratio Test.

Example 10.2. $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n^2} \rightsquigarrow a_n = \frac{(-1)^n 2^n}{n^2}$ and $a_{n+1} = \frac{(-1)^{n+1} 2^{n+1}}{(n+1)^2}$

Ratio Test:

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} 2^{n+1}/(n+1)^2}{(-1)^n 2^n/n^2} \right| = \lim_{n \rightarrow \infty} \left| \frac{-2n^2}{(n+1)^2} \right| = \lim_{n \rightarrow \infty} \frac{2n^2}{n^2 + 2n + 1} = 2 > 1$$

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n^2}$ is divergent by virtue of the Ratio Test.

THE ROOT TEST

- i. If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- ii. If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
- iii. If $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is inconclusive.
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Example 10.3. $\sum_{n=1}^{\infty} \left(\frac{5n+3}{7n+4}\right)^n$ $a_n = \left(\frac{5n+3}{7n+4}\right)^n$

Root Test: $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{5n+3}{7n+4} = \frac{5}{7} < 1$

$\therefore \sum_{n=1}^{\infty} \left(\frac{5n+3}{7n+4}\right)^n$ is absolutely convergent by virtue of the Root Test.

$\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$ $a_n = \left(\frac{-2n}{n+1}\right)^{5n}$

Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{-2n}{n+1}\right)^{5n}\right|} = \lim_{n \rightarrow \infty} \left(\frac{2n}{n+1}\right)^5 = 32 > 1$$

$\therefore \sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$ is divergent by virtue of the Root Test.

Example 10.4. $\sum_{n=1}^{\infty} \frac{n^{n+1}}{(2n+1)^n}$ $a_n = \frac{n^{n+1}}{(2n+1)^n}$

Root Test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n^{n+1}}{(2n+1)^n} \right|} \\ &= \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n \cdot n^n}{(2n+1)^n}} \\ &= \lim_{n \rightarrow \infty} \sqrt[n]{n} \cdot \left(\frac{n}{2n+1} \right) \\ &= \underbrace{\left(\lim_{n \rightarrow \infty} \sqrt[n]{n} \right)}_{1} \cdot \underbrace{\left(\lim_{n \rightarrow \infty} \frac{n}{2n+1} \right)}_{\frac{1}{2}} \end{aligned}$$

$$= \frac{1}{2} < 1 \quad \therefore \sum_{n=1}^{\infty} \frac{n^{n+1}}{(2n+1)^n} \text{ is absolutely convergent by virtue of the Root Test.}$$

Why is $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$?

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{n} &= \lim_{n \rightarrow \infty} n^{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} e^{\ln(n^{\frac{1}{n}})} \\ &= e^{\lim_{n \rightarrow \infty} \frac{\ln(n)}{n}} \quad \text{L'Hôpital's Rule} \\ &= e^{\lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1}} \\ &= e^0 = 1. \end{aligned}$$

STRATEGY FOR TESTING SERIES

Keep in mind all the tests that are available and the conditions needed to apply each test.

Example 10.5. $\sum_{m=1}^{\infty} \frac{m!}{m^m}$ $a_n = \frac{n!}{n^n}$ involves factorial \Rightarrow Ratio Test might work

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)! / (n+1)^{n+1}}{n! / n^n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(n+1) \cdot n^n}{(n+1)^{n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n} \\ &= e^{\lim_{n \rightarrow \infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}}} \\ &= e^{\lim_{n \rightarrow \infty} \frac{\left(\frac{n+1}{n}\right) \left(\frac{n+1-n}{(n+1)^2}\right)}{-\frac{1}{n^2}}} \end{aligned}$$

$$\begin{aligned} &= e^{\lim_{n \rightarrow \infty} \frac{-n}{n+1}} \\ &= e^{-1} < 1 \end{aligned}$$

$\therefore \sum_{m=1}^{\infty} \frac{m!}{m^m}$ is absolutely convergent by virtue of the Ratio Test.

Example 10.6. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

← all terms raised to power n^2 and it is not a geometric series
 ⇒ Root Test might work.

Root Test:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= \lim_{n \rightarrow \infty} \sqrt[n]{\left|\left(\frac{n}{n+1}\right)^{n^2}\right|} \\ &= \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n \\ &= \lim_{n \rightarrow \infty} e^{n \ln\left(\frac{n}{n+1}\right)} \\ &= e^{\lim_{n \rightarrow \infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}}} \quad \frac{0}{0} \\ &= e^{\lim_{n \rightarrow \infty} \frac{\left(\frac{n+1}{n}\right)\left(\frac{n+1-n}{(n+1)^2}\right)}{-\frac{1}{n^2}}} \\ &= e^{\lim_{n \rightarrow \infty} \frac{-n}{n+1}} \\ &= e^{-1} < 1 \end{aligned}$$

∴ $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ is absolutely convergent by virtue of the Root Test.

Example 10.7. $\sum_{k=1}^{\infty} k e^{-k^2}$

← $f(x) = x e^{-x^2}$ is easily integrated

⇒ Integral Test might work (if applicable)

Integral Test: Let $f(x) = x e^{-x^2}$

- $f(x)$ is continuous ✓ and positive ✓ for all x .
- $f'(x) = e^{-x^2} (1 - 2x^2) < 0$ for all $x > \frac{1}{\sqrt{2}}$
- ⇒ $f(x)$ is decreasing ✓ on $[1, \infty)$

$$\begin{aligned} \int_1^{\infty} x e^{-x^2} dx &= \lim_{t \rightarrow \infty} \int_1^t x e^{-x^2} dx \\ &= \lim_{t \rightarrow \infty} \left[-\frac{1}{2} e^{-x^2}\right]_1^t \\ &= \lim_{t \rightarrow \infty} -\frac{1}{2e^{t^2}} + \frac{1}{2e} \\ &= \frac{1}{2e} \text{ (hence convergent)} \end{aligned}$$

∴ $\sum_{n=1}^{\infty} n e^{-n^2}$ is convergent by virtue of The Integral Test.

Moreover...

$$\int_{n+1}^{\infty} x e^{-x^2} dx \leq R_n \leq \int_n^{\infty} x e^{-x^2} dx$$

by the Remainder Estimate Theorem...

Example 10.8. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin^2(n)}{n^2}$ ← this is an alternating series
 \Rightarrow AST might work.

$$b_n = \frac{\sin^2(n)}{n^2}$$

$$\text{Let } f(x) = \frac{\sin^2(x)}{x^2}.$$

$$\text{Then } f'(x) = \frac{2\sin(x)\cos(x) \cdot x^2 - 2x \cdot \sin^2(x)}{x^4} = \frac{2\sin(x)[x\cos(x) - 2\sin(x)]}{x^3}$$

But there is no x_0 for which $f'(x) < 0$ for all $x \geq x_0$

$\Rightarrow f(x)$ is never eventually decreasing

\Rightarrow part (i) of AST is not satisfied 😞

Consider $\sum |a_n|$ instead.

We have

$$0 \leq \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1} \sin^2(n)}{n^2} \right| = \sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (it's a p-series with $p=2 > 1$), we see

that $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1} \sin^2(n)}{n^2} \right|$ is convergent by virtue of the Comparison Test.

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin^2(n)}{n^2}$ is absolutely convergent, hence convergent.

STUDY GUIDE

Ratio Test Root Test General strategies for testing series

(Stewart, 8th ed.) §11.6 pg. 742: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 35, 37

§11.7 pg. 746: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37