9. Alternating Series & More Series Tests

- ◇ So far, several of the series tests we have require the terms of the series to be positive (*e.g. integral test, comparison test, limit comparison test*
- ◇ Now, we consider series with negative terms, specifically, so-called *alternating series*.

ALTERNATING SERIES

A series $\sum a_n$ in which consecutive terms have opposite signs is called an **alternating series**.

Ex. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

Ex. $0 - 1 + \sqrt{2} - \sqrt{3} + 2 - \sqrt{5} + \sqrt{6} - \sqrt{7} + \ldots = \sum_{n=0}^{\infty} (-1)^n \sqrt{n}$

Ex. $-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$

Suppose $\sum a_n$ is an alternating series. For all n, if we define $b_n = |a_n|$, then the nth term of the alternating series is of the form $a_n = (-1)^{n-1} b_n$ or $a_n = (-1)^n b_n$

Ex. For $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ we have $a_n = \frac{(-1)^{n-1}}{n}$ and $b_n = |a_n| = \frac{1}{n}$

ALTERNATING SERIES TEST (A.S.T.)

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \ldots$ (where $b_n > 0$ for all n) satisfies

i. $b_{n+1} \leq b_n$ for all n

and **ii.** $\lim_{n\to\infty} b_n = 0$, then the series is convergent.

* These notes are solely for the personal use of students registered in MAT1322.

If an alternating series $\sum(-1)^{n-1}b_n$ is convergent by virtue of the A.S.T., what happens?

$\hookrightarrow$ the partial sums $s_1, s_2, s_3, \dots$ will oscillate around the actual sum of the series $s = \sum(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6$

$s_1 = b_1$

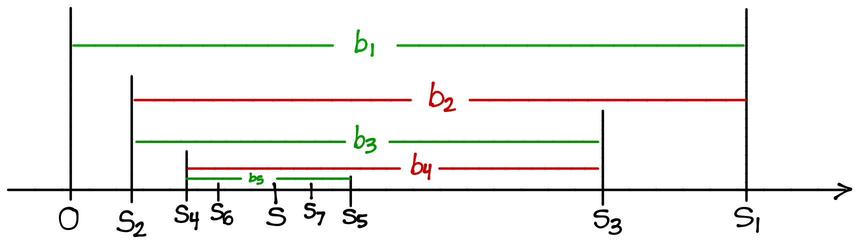
$s_2 = b_1 - b_2$

$s_3 = b_1 - b_2 + b_3$

$s_4 = b_1 - b_2 + b_3 - b_4$

$s_4 = b_1 - b_2 + b_3 - b_4 + b_5$

$\vdots$



Keep in mind: $b_1 \geq b_2 \geq b_3 \geq b_4 \geq b_5 \geq \cdots$ and $\lim\limits_{n\to\infty} b_n = 0$

Example 9.1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ $\leftarrow a_n = (-1)^{n-1}\frac{1}{n}$

$b_n = |a_n| = \frac{1}{n}$

i) $b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n$ for all n. ✓

ii) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$ ✓ $\therefore$ this series is convergent by virtue of the A.S.T.

$s_1 = 1 = 1$

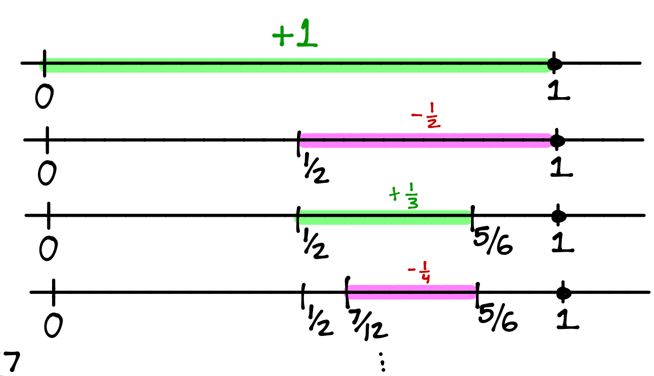
$s_2 = 1 - \frac{1}{2} = \frac{1}{2}$

$s_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

$s_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$

$s_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$

$\vdots$



In fact, later we will learn that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$

Example 9.2. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ ← this is an alternating series $a_n = \frac{(-1)^n n}{n^2+1}$ $b_n = \frac{n}{n^2+1}$

i.) Recall that $f(x) = \frac{x}{x^2+1}$ is a decreasing function on $(1, \infty)$

$\therefore b_{n+1} = f(n+1) \leq f(n) = b_n$ for all $n \geq 1$. ✓

ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = 0$ ✓

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ is convergent by virtue of the A.S.T.

Example 9.3. $\sum_{n=1}^{\infty} \frac{(-1)^n 2n}{5n+1}$ ← alternating series $a_n = \frac{(-1)^n 2n}{5n+1}$ $b_n = \frac{2n}{5n+1}$

<u>Note</u>: $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2n}{5n+1} = \frac{2}{5} \neq 0$ $\therefore$ AST is N/A.

However, we see that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n \cdot 2n}{5n+1}$ DNE

$\therefore$ this series is divergent by virtue of the Test for Divergence.

ALTERNATING SERIES ESTIMATION THEOREM (A.S.E.T.)

If $b_n > 0$ and $s = \sum(-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

i. $b_{n+1} \leq b_n$ and **ii.** $\lim_{n\to\infty} b_n = 0$

then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Example 9.4. $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$ How many terms do we need so that $|\text{error}| = |S - S_n| = |R_n| < 0.01$?

i.) $b_{n+1} = \frac{1}{(n+1)!} = \frac{1}{(n+1)\cdot n!} < \frac{1}{n!} = b_n$ ✓ ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n!} = 0$ ✓ $\Rightarrow$ convergent by A.S.T.

$\therefore$ by ASET $|R_n| \leq b_{n+1}$ $\Rightarrow$ we need n such that $b_{n+1} < 0.01 = \frac{1}{100}$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \cdots$$

Later, we will learn that $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1}$

Since $\frac{1}{5!} = \frac{1}{120} < 0.01$ we have $\left| \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} - \sum_{k=0}^{4} \frac{(-1)^k}{k!} \right| \leq \frac{1}{5!} < 0.01$

Example 9.5. Find the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ to within 0.005. $b_n = \frac{1}{n^2} > 0$

i.) $b_{n+1} = \frac{1}{(n+1)^2} < \frac{1}{n^2} = b_n$ for all n ✓

ii.) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$ ✓

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is convergent by virtue of the A.S.T.

Moreover, $|R_n| = |s - s_n| \leq b_{n+1}$ by virtue of the A.S.E.T.

Ex $|R_6| = |b_7 - b_8 + b_9 - b_{10} + \dots| = \left| \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} - \frac{1}{10^2} + \dots \right| \leq \frac{1}{7^2}$

In order for s_n to be within 0.005 of the sum $s = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$
it suffices to have $b_{n+1} \leq 0.005$

$\Rightarrow$ solve for n such that $b_{n+1} \leq 0.005 = \frac{1}{200}$

$$\Rightarrow \frac{1}{(n+1)^2} \leq 0.005 = \frac{1}{200}$$

$$\Rightarrow 200 \leq (n+1)^2$$

$$\Rightarrow 14.1421\ldots \leq n+1$$

$$\Rightarrow 14 \leq n$$

Now $S_{14} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots - \frac{1}{14^2} \approx 0.84$ and $b_{15} = \frac{1}{15^2} \approx 0.00\overline{4}$

Thus, $|s - s_{14}| \leq b_{15} = 0.00\overline{4}$ (< 0.005)

Equivalently, $s_{14} - b_{15} \leq s \leq s_{14} + b_{15}$

$$\therefore 0.84\ldots - 0.00\overline{4} \leq s \leq 0.84\ldots + 0.00\overline{4}$$

◇ A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent $\quad |a_n| = \frac{1}{n^2}$

Theorem. If a series $\sum a_n$ is absolutely convergent, then it is convergent.

proof For all n, we have $0 \le a_n + |a_n| \le 2|a_n|$.

If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ is convergent.

Consequently, $\sum 2|a_n| = 2\sum |a_n|$ is also convergent.

By the Comparison Test, it follows that $\sum (a_n + |a_n|)$ is convergent

Now, $\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$ is the difference of two convergent series, hence $\sum a_n$ must be convergent too.

∎

◇ A series $\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is divergent while the series itself $\sum_{n=1}^{\infty} a_n$ is convergent.

Example 9.6. Discuss the convergence of the alternating harmonic series. Is it divergent? If not, is it absolutely convergent, or is it conditionally convergent?

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is convergent by virtue of the Alternating Series Test

but the series of its absolute values is $\sum_{n=1}^{\infty} \frac{1}{n}$ (the harmonic series) which we know is divergent (p-series, p=1)

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent.

Example 9.7. Is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ divergent? Is it absolutely or conditionally convergent?

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ is convergent by virtue of the Alternating Series Test (verify this!)

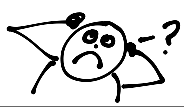
but the series of its absolute values is $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is divergent by virtue of The Integral Test. (see Example 7.3)

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ is conditionally convergent.

Example 9.8. Show that the series $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(3n)}{3^n}$ is convergent. Note $a_n = \dfrac{\sin(3n)}{3^n}$

this is not an alternating series, but its terms are not all positive.

• we cannot apply AST	• we cannot apply The Integral Test • we cannot apply Comparison Test • we cannot apply Limit Comparison Test



Consider the series $\displaystyle\sum_{n=0}^{\infty} \left|\frac{\sin(3n)}{3^n}\right|$ ← this series has positive terms.

We have

$0 \le \left|\dfrac{\sin(3n)}{3^n}\right| \le \dfrac{1}{3^n}$ for all $n \ge 1$ $\therefore 0 \le \displaystyle\sum_{n=0}^{\infty} \left|\frac{\sin(3n)}{3^n}\right| \le \sum_{n=0}^{\infty} \frac{1}{3^n}$

since $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n}$ is a geometric series with $|r| = \left|\frac{1}{3}\right| < 1$, it's convergent

$\therefore \displaystyle\sum_{n=0}^{\infty} \left|\frac{\sin(3n)}{3^n}\right|$ is convergent by Comparison.

$\therefore \displaystyle\sum_{n=0}^{\infty} \frac{\sin(3n)}{3^n}$ is absolutely convergent hence is convergent.

STUDY GUIDE

- ☐ **Alternating Series Test**
- ☐ **Absolutely Convergent**
- ☐ **Alternating Series Estimation Theorem**
- ☐ **Conditionally Convergent**

Exercises (Stewart, 8th ed.)	§**11.5** pg. 736:	1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 23, 25, 27, 29
	§**11.6** pg. 742:	1, 3, 5, 31, 33, 37, 51