DGD 11

(1) $z = xy^2 - x^2 y$, $x = t^2+1$, $y = t^2-1$

Find $\frac{dz}{dt}$

Solution: Here $t \longmapsto (x,y) \longmapsto z$, whence

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (y^2 - 2xy)\cdot 2t + (2xy - x^2)\cdot 2t$$

$$= 2t\,(y^2 - x^2) = 2t(y-x)(y+x)$$

$$= 2t\,(-2)\cdot 2t^2$$

$$= -8t^3$$

② $z=(x-y)^5$, $x=s^2t$, $y=st^2$

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

<u>Solution:</u> Here $(s,t)\mapsto(x,y)\longrightarrow z$, where

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$=5(x-y)^4\cdot 2st-5(x-y)^4\cdot t^2$$

$$=5t(x-y)^4(2s-t)=5t(s^2t-st^2)^4(2s-t)$$

$$=5s^4t^5(s-t)^4(2s-t)$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$=5(x-y)^4\cdot s^2-5(x-y)^4\cdot 2st$$

$$=5s(x-y)^4(s-2t)=5s(s^2t-st^2)^4(s-2t)$$

$$=5s^5t^4(s-t)^4(s-2t)$$

③ $z = x^3 + x^2 y$, $x = s + 2t - u$, $y = stu^2$

Find $\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}, \frac{\partial z}{\partial u}$ for $s=2, t=-1, u=1$

Solution: Here $(s,t,u) \longrightarrow (x,y) \longrightarrow z$, and

$x(2,-1,1) = -1$, $y(2,-1,1) = -2$, whence

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (3x^2+2xy) + x^2 \cdot tu^2, \text{ and}$$

$$\frac{\partial z}{\partial s}\Big|_{(s,t,u)=(2,-1,1)} = (3+4) + (-1)^2 \cdot (-1) \cdot 1^2 = 7-1 = 6$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (3x^2+2xy)\cdot 2 + x^2 \cdot su^2, \text{ and}$$

$$\frac{\partial z}{\partial t}\Big|_{(s,t,u)=(2,-1,1)} = (3+4)\cdot 2 + (-1)^2 \cdot 2 \cdot 1^2 = 14 + 2 = 16$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (3x^2+2xy)(-1) + x^2 \cdot 2stu, \text{ and}$$

$$\frac{\partial z}{\partial u}\Big|_{(s,t,u)=(2,-1,1)} = (3+4)\cdot(-1) + (-1)^2 \cdot 2 \cdot 2(-1)\cdot 1$$

$$= -7 - 4 = -11$$

④ If $z = f(x,y)$, $x = r^2+s^2$, $y = 2rs$, find $\dfrac{\partial^2 z}{\partial r \partial s}$

Solution: Here $(r,s) \mapsto (x,y) \mapsto z$, so that

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = f_x \cdot 2r + f_y \cdot 2s$$

$$\frac{\partial^2 z}{\partial r \partial s} = \frac{\partial\left(\frac{\partial z}{\partial r}\right)}{\partial s} = \frac{\partial\left(f_x \cdot 2r + f_y \cdot 2s\right)}{\partial s}$$

$$= 2\frac{\partial f_x}{\partial s}\cdot r + 2\frac{\partial f_y}{\partial s}\cdot s + 2f_y$$

Now, in the same way as above, for any function $g = g(x,y)$

$$\frac{\partial g}{\partial s} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial s} = g_x \cdot 2s + g_y \cdot 2r, \text{ where}$$

$$\frac{\partial^2 z}{\partial r \partial s} = 2\left(f_{xx}\cdot 2s + f_{xy}\cdot 2r\right)\cdot r$$

$$+ 2\left(f_{yx}\cdot 2s + f_{yy}\cdot 2r\right)\cdot s + 2f_y$$

$$= 2f_y + 4sr\left(f_{xx} + f_{yy}\right) + 4(r^2+s^2)f_{xy}$$

The following exercises are based on Lecture 23 (to be given after D&D 11, so that the definitions from Lecture 23 are additionally explained in the solutions)

5) Find the gradient of the function $f(x,y) = x^2 \ln y$ and evaluate it at $P=(3,1)$

Solution: The gradient vector of a function f at a point (x,y) is $\nabla f(x,y) = (f_x(x,y), f_y(x,y))$

Here $f_x = 2x \ln y$, $f_y = \frac{x^2}{y}$, so that

$$\nabla f(x,y) = \left(2x \ln y, \frac{x^2}{y}\right), \text{ and}$$

$$\nabla f(3,1) = (0, 9)$$

⑥ Find the gradient of the function

$f(x,y,z) = x^2yz - xyz^3$ at $P=(2,-1,1)$

Solution: In the same for a function of 3 variables $f(x,y,z)$ its gradient is

$$\nabla f(x,y,z) = \left(f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\right)$$

Thus, for the given function

$$\nabla f(x,y,z) = \left(2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2\right)$$

so that

$$\nabla f(2,-1,1) = (-4+1, 4-2, -4+3\cdot 2)$$
$$= (-3, 2, 2)$$

(7) Find the directional derivative of the function $f(x,y)=e^x \sin y$ at the point $\left(0, \frac{\pi}{3}\right)$ in the direction of the vector $\bar{v}=(-3,4)$

<u>Solution:</u> One has to normalize the vector $\bar{v}$ first; since $|\bar{v}|=\sqrt{3^2+4^2}=5$, we obtain $\bar{u}=\frac{\bar{v}}{|\bar{v}|}=\left(-\frac{3}{5}, \frac{4}{5}\right)$. Now, the directional derivative at point (x,y) is

$$D_{\bar{u}} f(x,y) = \bar{u} \cdot \nabla f(x,y)$$

$$= \left(-\frac{3}{5}, \frac{4}{5}\right)\left(e^x \sin y, e^x \cos y\right), \text{ whence}$$

$$D_{\bar{u}} f\left(0, \frac{\pi}{3}\right) = \left(-\frac{3}{5}, \frac{4}{5}\right)\left(e^x \sin y, e^x \cos y\right)\Big|_{\substack{x=0 \\ y=\frac{\pi}{3}}}$$

$$= \left(-\frac{3}{5}, \frac{4}{5}\right)\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$$= \frac{-3\sqrt{3}+4}{10}$$

(8) Find the directional derivative of the function $f(x,y,z) = x^2y + y^2z$ at the point $(1,2,3)$ in the direction of the vector $\bar{v} = (2,-1,2)$

Solution: In the same way is for 2 variables, one has to normalize the vector $\bar{v}$ first, by passing to

$$\bar{u} = \frac{\bar{v}}{\|\bar{v}\|} = \frac{(2,-1,2)}{\sqrt{4+1+4}} = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$$

The the directional direvative is

$$D_{\bar{u}} f(x,y,z) = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) \cdot \nabla f(x,y,z)$$

$$= \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)\left(2xy, x^2+2yz, y^2\right),$$ whence

$$D_{\bar{u}} f(1,2,3) = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)\left(4, 13, 4\right) = 1$$

(9) Find the equation of the tangent plane to the surface $z = 2x^2 + y^2 - 5y$ at the point $(1, 2, -4)$

Solution: If $f(x_0, y_0) = z_0$, then the point (x_0, y_0, z_0) belong to the surface $z = f(x, y)$, and the tangent plane to the surface at this point is

$$z - z_0 = \nabla f(x_0, y_0)(x - x_0, y - y_0), \text{ i.e.}$$

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

In our case

$$f_x(x_0, y_0) = f_x(1, 2) = 4x \Big|_{\substack{x=1 \\ y=2}} = 4$$

$$f_y(x_0, y_0) = f_y(1, 2) = (2y - 5)\Big|_{\substack{x=1 \\ y=2}} = -1$$

whence

$$z = z_0 + 4(x - x_0) - (y - y_0)$$

$$= -4 + 4(x - 1) - (y - 2)$$

$$= 4x - y - 6$$