DGD 10

① Solve the differential equation $\frac{dy}{dx} = y(1-y)$ with the initial condition

$$y(0) = \frac{1}{2}$$

Solution. Separate the variables:

$$\int \frac{dy}{y(1-y)} = \int dx \Rightarrow \int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \int dx \Rightarrow$$

$$\ln|y| - \ln|1-y| = x + C \Rightarrow \ln\left|\frac{y}{1-y}\right| = x + C$$

$$\Rightarrow \left|\frac{y}{1-y}\right| = e^{x+C} \quad \text{or} \quad \left|\frac{1-y}{y}\right| = \left|\frac{1}{y} - 1\right| = e^{-x-C}$$

$$\Rightarrow \frac{1}{y} - 1 = Ke^{-x} \quad \text{with } K = \pm e^{-C}$$

$$\Rightarrow \frac{1}{y} = 1 + Ke^{-x} \Rightarrow y = \frac{1}{1+Ke^{-x}}$$

is the general solution

Then for $x=0$ $y = \frac{1}{1+K} = \frac{1}{2}$, whence $K=1$, so that the solution satisfying the initial conditions is

$$y(x) = \frac{1}{1+e^{-x}}$$

(2) Solve $\frac{dy}{dx} = 3x^2y^2$ with $y(0) = \frac{1}{2}$

<u>Solution</u> $\int \frac{dy}{y^2} = \int 3x^2\,dx \Rightarrow -\frac{1}{y} = x^3 + C$

$\Rightarrow$ $y = -\frac{1}{x^3+C}$ is the general solution

Since $y(0) = \frac{1}{2}$, $\frac{1}{2} = -\frac{1}{C} \Rightarrow C = -2$

Therefore, the solution satisfying the initial conditions is

$$y = -\frac{1}{x^3-2}$$

(3) Solve $\frac{dy}{dx} = x e^{y}$, $y(0) = 0$

Solution $e^{-y} dy = x\, dx \implies$

$$-e^{-y} = \frac{x^2}{2} + C \implies$$

$$e^{-y} = -\frac{x^2}{2} - C$$

$$y = -\ln\left(-\frac{x^2}{2} - C\right)$$

is the general solution

For $x = 0$ $y = -\ln(-C) = 0$

$\implies C = -1$, so that

$$y = -\ln\left(-\frac{x^2}{2} + 1\right)$$

(4) Solve $\frac{dx}{dt} = x^2 \ln t$, $x(1) = -1$

Solution: $\int \frac{dx}{x^2} = \int \ln t \, dt \implies$

$$-\frac{1}{x} = t\ln t - t + C$$

(integration by parts $\int uv' = uv - \int u'v$ with $u = \ln t$, $v = t$) $\implies$

$$x(t) = -\frac{1}{t\ln t - t + C}$$

is the general solution

For $t = 1$ $x(1) = -\frac{1}{C-1} = -1$,

whence $C = 2$, so that

$$x(t) = -\frac{1}{t\ln t - t + 2}$$

⑤ Solve $\frac{dy}{dx} = 1 + x + y + xy$

with $y(-2) = 1$

<u>Solution</u> $\frac{dy}{dx} = (1+x)(1+y) \Rightarrow$

$$\frac{dy}{1+y} = (1+x)\,dx$$

$$\ln(1+y) = x + \frac{x^2}{2} + c$$

$$1 + y = e^{x + \frac{x^2}{2} + c}$$

$$y = K e^{x + \frac{x^2}{2}} - 1, \quad K = e^c$$

is the general solution

For $x = -2$

$$y(-2) = K e^0 - 1 = K - 1 = 1,$$

whence $K = 2$, so that

$$y = 2 e^{x + \frac{x^2}{2}} - 1$$

⑥ Find the first partial derivatives of $f(x,y,z) = x^3yz^2 + 2yz$

Solution

$f_x = 3x^2yz^2 + 2yz$

$f_y = x^3z^2 + 2z$

$f_z = 2x^3yz + 2y$

(7) Find the first partial derivatives of $w(u,v) = \dfrac{e^v}{u+v^2}$

Solution:

$$w_u = -\frac{e^v}{(u+v^2)^2}$$

$$w_v = \frac{e^v(u+v^2) - e^v \cdot 2v}{(u+v^2)^2}$$

$$= \frac{(u+v^2-2v)\,e^v}{(u+v^2)^2}$$

8) Find all the second partial derivatives of $f(x,y) = x^4 y - x \ln y$

Solution: $f_x = 4x^3 y - \ln y$

$f_y = x^4 - \frac{x}{y}$, whence

$f_{xx} = (f_x)_x = 12x^2 y$

$f_{yy} = (f_y)_y = \frac{x}{y^2}$

The mixed derivative f_{xy} can be found both as

$f_{xy} = (f_x)_y = (4x^3 y - \ln y)_y = 4x^3 - \frac{1}{y}$

and

$f_{xy} = (f_y)_x = \left(x^4 - \frac{x}{y}\right)_x = 4x^3 - \frac{1}{y}$

(9) Show that the function

$u(x,t) = \sin x \sin t$

is a solution of the equation $u_{tt} - u_{xx} = 0$

Solution: $u_t = \sin x \cos t$,

$u_{tt} = -\sin x \sin t$,

and

$u_x = \cos x \sin t$

$u_{xx} = -\sin x \sin t$,

whence $u_{tt} - u_{xx} = 0$