ODO 9

① Find the Maclaurin series for the function $f(x) = x \sin x$

Solution. We begin with $g(x) = \sin x$ with $g(0) = 0$. Then $g'(x) = \cos x$, $g'(0) = 1$;
$g''(x) = -\sin x$, $g''(0) = 0$,
$g'''(x) = -\cos x$, $g'''(0) = -1$,
$g^{(4)}(x) = \sin x$, $g^{(4)}(0) = 0$, and
the sequence further repeats with period 4.

Therefore, the Maclaurin series for g is
$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

By multiplying it by x, we obtain the Maclaurin series for $f(x) = x\,g(x)$:
$$x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots$$

② Find the Maclaurin series for

$$f(x) = (1+2x)^{1/3}$$

Solution: Begin with $f(0) = 1$.

Further,

$$f'(x) = \frac{2}{3}(1+2x)^{-2/3}, \quad f'(0) = \frac{2}{3}$$

$$f''(x) = -\frac{8}{9}(1+2x)^{-5/3}, \quad f''(0) = -\frac{8}{9}$$

$$f'''(x) = \frac{80}{27}(1+2x)^{-8/3}, \quad f'''(0) = \frac{80}{27}$$

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Therefore, the series is

$$1 + \frac{2}{3}x - \frac{4}{9}x^2 + \frac{40}{81}x^3 - \dots$$

(3) Find the Maclaurin series for

$$f(x) = \frac{1}{\sqrt{1-x}}$$

Solution $\quad f(0) = 1,$

$$f'(x) = \left((1-x)^{-1/2}\right)' = \frac{1}{2}(1-x)^{-3/2}$$

$$f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{3}{4}(1-x)^{-5/2}, \quad f''(0) = \frac{3}{4},$$

$$f'''(x) = \frac{15}{8}(1-x)^{-7/2}, \quad f'''(0) = \frac{15}{8}$$

Therefore, the Maclaurin series is

$$1 + \frac{x}{2} + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots$$

(4) Evaluate $\int_0^t e^{-x^2}\,dx$ as an infinite series

<u>Solution</u>. We use the Maclaurin expansion

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots,$$

whence

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots,$$

whence

$$\int_0^t e^{-x^2}\,dx = \int_0^t \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots\right)$$

$$= t - \frac{t^3}{3} + \frac{t^5}{5\cdot 2} - \frac{t^7}{7\cdot 3!} + \frac{t^9}{9\cdot 4!} - \cdots$$

5 Evaluate $\displaystyle\lim_{x\to 0} \frac{e^x-1-x}{x^2} = L$

<u>Solution</u>: We use the Maclaurin expansion

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

whence

$$\frac{e^x-1-x}{x^2} = \frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots,$$

whence $L = \frac{1}{2}$

(6) Show that $y(x) = \frac{2}{3}e^{x} + e^{-2x}$ is a solution of the differential equation

$$y' + 2y = 2e^{x}$$

Solution: $y' = \frac{2}{3}e^{x} - 2e^{-2x}$,

whence

$$y' + 2y = \frac{2}{3}e^{x} - 2e^{-2x} + \frac{4}{3}e^{x} + 2e^{-2x}$$

$$= 2e^{x}$$

(7) Verify that $y(t) = -t\cos t - t$ is a solution of the equation $ty' = y + t^2 \sin t$ with the initial condition $y(\pi) = 0$

<u>Solution</u> $y'(t) = -\cos t + t\sin t - 1$,

whence $ty' = -t\cos t + t^2 \sin t - t$

$$= y + t^2 \sin t;$$

it remains to check the initial condition:

$$y(\pi) = -\pi\cos\pi - \pi = \pi - \pi = 0$$

(8) Which of the following functions are solutions of the differential equation $y'' + y = \sin x$: (a) $y = \sin x$

(b) $y = \cos x$ (c) $y = \frac{1}{2} x \sin x$ (d) $y = -\frac{1}{2} x \cos x$

<u>Solution</u>: (a): $y''(x) = -\sin x$,

whence $y'' + y = 0$

(b): $y''(x) = -\cos x \Longrightarrow y'' + y = 0$

(c): $y'(x) = \frac{1}{2}(\sin x + x \cos x)$

$$y''(x) = \frac{1}{2}(\cos x + \cos x - x \sin x)$$

$$= \cos x - \frac{1}{2} x \sin x$$

$$\Longrightarrow y'' + y = \cos x$$

(d) $y'(x) = \frac{1}{2}(-\cos x + x \sin x)$

$$y''(x) = \frac{1}{2}(2 \sin x + x \cos x)$$

$$\Longrightarrow y + y'' = \sin x$$

Thus, it is only (d) that is a solution of the diff. equation