DGD 8

① Find the radius of the convergence and the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n} (x-1)^n$$

Solution: Put $a_n = \frac{(-1)^n}{(2n-1)2^n}(x-1)^n$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x-1}{2}\right| \cdot \frac{2n-1}{2n+1} \longrightarrow \left|\frac{x-1}{2}\right| = L, \text{ so}$$

that $L < 1 \iff |x-1| < 2$

Therefore, the radius of convergence is $R = 2$.

Since the interval of convergence is centered at 1, its endpoints are $x_1 = 1-2 = -1$ and $x_2 = 1+2 = 3$.

For $x = -1$ $\quad a_n = \frac{1}{2n-1}$, so that the series diverges

For $x = 3$ $\quad a_n = \frac{(-1)^n}{2n-1}$, and the series converges

Thus, the convergence interval is $[-1, 3]$

② Find the radius of the convergence and the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$$

Solution: Write $a_n = \frac{(2x-1)^n}{5^n \sqrt{n}} = \left(\frac{2}{5}\right)^n \frac{1}{\sqrt{n}} \left(x-\frac{1}{2}\right)^n$

Then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{5}\left|x-\frac{1}{2}\right| \frac{\sqrt{n}}{\sqrt{n+1}} \longrightarrow \frac{2}{5}\left|x-\frac{1}{2}\right| = L$

so that $L < 1 \iff \left|x-\frac{1}{2}\right| < \frac{5}{2}$

Therefore, the radius of convergence is $R = \frac{5}{2}$, and the endpoints of the interval of convergence are $x_{1,2} = \frac{1}{2} \pm \frac{5}{2}$, i.e. $x_1 = -2$ and $x_2 = 3$. For $x = -2$ $a_n = \frac{(-5)^n}{5^n} \frac{1}{\sqrt{n}} = \frac{(-1)^n}{\sqrt{n}}$, so that the series converges, whereas for $x = 3$ $a_n = \frac{1}{\sqrt{n}}$, and the series diverges. Therefore, the convergence interval is $[-2, 3)$.

(3) Find a power representation for the function and determine the interval of convergence: $f(x) = \dfrac{x^2}{2x^4+1}$

Solution. $f(x) = x^2 \cdot \dfrac{1}{1+2x^4} = x^2 \cdot \dfrac{1}{1-(-2x^4)}$

$= x^2(1 - 2x^4 + 4x^8 - 8x^{12} + \ldots)$

$= x^2 - 2x^6 + 4x^{10} - 8x^{12}$,

which converges for $|2x^4| < 1$, i.e. $x \in \left(-\dfrac{1}{2^{1/4}}, \dfrac{1}{2^{1/4}}\right)$. At the endpoints $2x^4 = 1$, so that the series diverges. Finally, the convergence interval is $\left(-\dfrac{1}{2^{1/4}}, \dfrac{1}{2^{1/4}}\right)$.

(4) Find a power representation for the function and determine the interval of convergence: $f(x) = \ln(2-x)$

Solution. Use the fact that

$$[\ln(2-x)]' = -\frac{1}{2-x} = -\frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}}$$

$$= -\frac{1}{2}\left[1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \cdots\right]$$

By integrating this series we obtain

$$f(x) = C - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{3\cdot 2^3} - \frac{x^4}{4\cdot 2^4} - \cdots$$

$$= C - \sum_{n=1}^{\infty} \frac{x^n}{n\cdot 2^n}$$

By taking $x=0$ we obtain $C = \ln 2$

By using the ratio test we obtain $R=2$ (cf. Ex 1, 2), so that the endpoints of the interval of convergence are ± 2. At $x=-2$ the series

$C - \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, whereas

for $x=2$ the series $C - \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so that the convergence interval is $[-2, 2)$

5. Find a power representation for the function and determine the interval of convergence: $f(x) = \ln\left(\frac{1+x}{1-x}\right)$

Solution

$$f'(x) = [\ln(1+x) - \ln(1-x)]' = \frac{1}{1+x} + \frac{1}{1-x}$$

$$= (1 - x + x^2 - x^3 + \dots) + (1 + x + x^2 + \dots)$$

$$= 2(1 + x^2 + x^4 + \dots), \text{ which converges}$$

for $|x^2| < 1$, i.e., $x \in (-1, 1)$,

By integrating,

$$f(x) = C + 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \dots$$

Since $f(0) = \ln 1 = 0$, $C = 0$.

We have already found the endpoints $x = \pm 1$ of the convergence interval. For $x = -1$ the series $-2 - \frac{2}{3} - \frac{2}{5} - \frac{2}{7} - \dots$ diverges, and the same happens for $x = 1$, whence the convergence interval is $(-1, 1)$.

(6) Use differentiation to find a power series for $f(x) = \frac{1}{(1+x)^2}$

Solution. $\int f(x)\,dx = -\frac{1}{1+x} + C$

Put $F(x) = -\frac{1}{1+x}$, so that $F'(x) = f(x)$

Since

$$F(x) = -\frac{1}{1+x} = -(1 - x + x^2 - x^3 + \dots)$$

$$= -1 + x - x^2 + x^3 - x^4 + \dots,$$

by differentiating the above series we obtain

$$f(x) = 1 - 2x + 3x^2 - 4x^3 + \dots$$

(7) Evaluate the indefinite integral as a power series. What is the radius of convergence?

$$I = \int \frac{t}{1-t^3}\, dt$$

Solution We have

$$f(t) = \frac{t}{1-t^3} = t\left(1+t^3+t^6+t^9+\ldots\right)$$

$$= t + t^4 + t^7 + t^{10} + \ldots,$$

whence

$$\int f(t)\, dt = C + \frac{t^2}{2} + \frac{t^5}{5} + \frac{t^8}{8} + \ldots$$

The radius of converges of this series is the same as for $f(t)$, which converges for $|t^3| < 1$, i.e. $R = 1$

8) Evaluate the indefinite integral as a power series. What is the radius of convergence?

$$I = \int x^2 \ln(1+x)\, dx$$

Solution. Since $(\ln(1+x))' = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$

$$\ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

For $x = 0$ $\ln(1+x) = 0$, whence $C = 0$

Thus,

$$x^2 \ln(1+x) = x^3 - \frac{x^4}{2} + \frac{x^5}{3} - \frac{x^6}{4} + \ldots,$$

whence

$$I = C + \frac{x^4}{4} - \frac{x^5}{2\cdot 5} + \frac{x^6}{3\cdot 6} - \frac{x^7}{4\cdot 7} + \ldots$$

Since the convergence radius is preserved under integration, it is the same as for the original series $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$, i.e., $R = 1$